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THE

# THEORY OF ERRORS

AND

# METHOD OF LEAST SQUARES

BY

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# PREFACE.

THE basis adopted in this book for the theory of accidental errors is that laid down by Gauss in the *Theoria Motus Corporum Cœlestium* (republished as vol. vii of the *Werke*), which may be described for the most part in his own words, as follows :

"The hypothesis is in fact wont to be considered as an axiom that, if any quantity has been determined by several direct observations, made under similar circumstances and with equal care, the arithmetical mean between all the observed values presents the most probable value, if not with absolute rigor, at least very nearly so, so that it is always safest to adhere to it." (Art. 177.)

Then introducing the notion of a law of facility of error to give precise meaning to the phrase "most probable value," we cannot do better than to adopt that law of facility in accordance with which the arithmetical mean is the most probable value. After deriving this law and showing that it leads to the principle of least squares, he says : "This principle, which in all applications of mathematics to natural philosophy admits of very frequent use, ought everywhere to hold good as an axiom by the same right as that by which the arithmetical mean between several observed values of the same quantity is adopted as the most probable value." (Art. 179.)

Accordingly no attempt has been made to demonstrate the principle of the arithmetical mean, nor to establish the exponential law of facility by any independent method. It has been deemed important, however, to show the self-consistent nature of the law, in the fact that its assumption for the errors of direct observation involves as a consequence a law of the same form for any linear function of observed quantities, and particularly for the final determination which results from our method. This persistence in the form of the law has too frequently been assumed, in order to simplify the demonstrations ; but at the expense of soundness.

No place has been given to the so-called criteria for the rejection of doubtful observations. Any doubt which attaches to an observation on account of the circumstances under which it is made, is recognized, in the practice of skilled observers, in its rejection, or in assigning it a small weight at the time it is made ; but these criteria profess to justify the subsequent rejection of an observation on the ground that its residual is found to exceed a certain limit. With respect to this Professor Asaph Hall says: "When observations have been honestly made I dislike to enter upon the process of culling them. By rejecting the large residuals the work is made to appear more accurate than it really is, and thus we fail to get the right estimate of its quality." (*The Orbit of Iapetus*, p. 40, *Washington Observations for* 1882, *Appendix I.*)

The notion that we are entitled to reject an observation, that is, to give it no weight, when its residual exceeds a certain limit, would seem to imply that we ought to give less than the usual weight to those observations whose residuals fall just short of this limit, in fact that we ought to revise the observations, assigning weights which diminish as the residuals increase. Such a process might appear at first sight plausible,

but it would be equivalent to a complete departure from the principle of the arithmetical mean and the adoption of a new law of facility. For this we have no justification, either from theory or from the examination of the errors of extended sets of observations.

In the discussion of Gauss's method of solving the normal equations, the notion of the 'reduced observation equations' (see Arts. 154, 155) which gives a new interpretation to the 'reduced normal equations' has been introduced with advantage. This conception, although implied in Gauss's elegant discussion of the sum of the squares of the errors (see Art. 160), seems not to have appeared explicitly in any treatise prior to the third edition of W. Jordan's *Handbuch der Vermessungskunde* (Stuttgart, 1888). To this very complete work, and to Oppolzer's *Lehrbuch zur Bahnbestimmung der Kometen und Planeten*, I am indebted for the forms recommended for the computations connected with Gauss's method, and for many of the examples.

W. W. J.

U. S. NAVAL ACADEMY, June, 1892.

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# THE THEORY OF ERRORS AND METHOD OF LEAST SQUARES.

## I.

### Introductory.

### *Errors of Observation.*

**1.** A quantity of which the magnitude is to be determined is either directly measured, or, as in the more usual case, deduced by calculation from quantities which are directly measured. The result of a direct measurement is called an *observation*. Observations of the kind here considered are thus of the nature of readings upon some scale, generally attached to an instrument of observation. The *least count* of the instrument is the smallest difference recognized in the readings of the instrument, so that every observation is recorded as an integral multiple of the least count.

**2.** Repeated observations of the same quantity, even when made with the same instrument and apparently under the same circumstances, will nevertheless differ materially. An increase in the nicety of the observations, and the precision of the instrument, may decrease the discrepancies in actual magnitude; but at the same time, by diminishing the least count, their numerical measures will generally be increased; so that, with the most refined instruments, the discrepancies may amount to many times the least count. Thus every observation is subject to an *error*, the error being the difference between the observed value and the true value; an observed value which exceeds the true value is regarded as having a positive error, and one which falls short of it as having a negative error.

3. An error may be regarded as the algebraic sum of a number of elemental errors due to various causes. So far as these causes can be ascertained, their results are not errors at all, in the sense in which the term is here used, and are supposed to have been removed by means of proper corrections. *Systematic errors* are such as result from unknown causes affecting all the observations alike. These again are not the subjects of the "theory of errors," which is concerned solely with the *accidental errors* which produce the discrepancies between the observations.

### *Objects of the Theory.*

4. It is obvious that when a set of repeated observations of the same quantity are made, the discrepancies between them enable us to judge of the degree of accuracy we have attained. Speaking in general terms, of two sets of observations, that is the best which exhibits upon the whole the smaller discrepancies. It is obvious also that from a set of observations we shall be able to obtain a result in which we can have greater confidence than in any single observation.

It is one of the objects of the theory of errors to deduce from a number of discordant observations (supposed to be already individually corrected, so far as possible) the best attainable result, together with a measure of its accuracy; that is to say, of the degree of confidence we are entitled to place in it.

5. When a number of unknown quantities are to be determined by means of equations involving observed quantities, the quantities sought are said to be *indirectly observed.* It is necessary to have as many such *observation equations* as there are unknown quantities. The case considered is that in which it is impossible to make repeated observations of the individual observed elements of the equations. These may, for example, be altitudes or other astronomical magnitudes which vary with the time, so that the corresponding times are also among the observed quantities. Nevertheless, there is the same advantage in employing a large number of observation equations that there

is in the repetition of direct observations upon a single required quantity. If there are $n$ unknown quantities, any group containing $n$ of the equations would determine a set of values for the unknown quantities; but these values would differ from those given by any other group of $n$ of the equations.

We may now state more generally the object of the theory of errors to be, when given more than $n$ observation equations involving $n$ unknown quantities, the equations being somewhat inconsistent, to derive from them the best determination of the values of the several unknown quantities, together with a measure of the degree of accuracy attained.

**6.** It will be noticed that, putting $n = 1$, this general statement includes the case of *direct observations*, in which all the equations are of the form

$$X = x_1, \quad X = x_2, \quad \ldots,$$

where $X$ is the quantity to be determined, and each equation gives an independent statement of its value.

We commence with this case of direct observations of a single quantity, and our first consideration will be that of the best determination which can be obtained from a number of such observations.

## II.

### Independent Observations of a Single Quantity.

*The Arithmetical Mean.*

**7.** Whatever rule we adopt for deducing the value to be accepted as the final result derived from several independent observations, it must obviously be such that when the observations are equal the result shall be the same as their common value. When the observations are discordant, such a rule produces an intermediate or *mean* value. Thus, if there be $n$ quantities, $x_1, x_2, \ldots x_n$, the expressions

$$\frac{\Sigma x}{n}, \quad \sqrt[n]{(x_1 x_2 \ldots x_n)}, \quad \sqrt{\frac{\Sigma x^2}{n}}, \quad \text{etc.},$$

give different sorts of mean values. Of these, the one first written, which is the *arithmetical mean*, is the simplest, and it is also that which has universally been accepted as the final value when $x_1, x_2, \ldots x_n$ are independently observed values of a single quantity $x$, the observations being all supposed equally good.

*Residuals.*

**8.** The differences between the several observed values and the value which we take as our final determination of the true value are called the *residuals* of the observations. The residuals are then what we take to be the errors of the observations; but they differ from them, of course, by the amount of error existing in our final determination. If the observed values were laid down upon a straight line, as measured from any origin, the residuals would be the abscissas of the points thus representing the observations when the point corresponding to the final value adopted is taken as the origin.

**9.** *In the case of the arithmetical mean, the algebraic sum of the residuals is zero.* For, if $a$ denote the arithmetical mean of the $n$ quantities $x_1, x_2, \ldots x_n$, we have

$$a = \frac{\Sigma x}{n}, \quad \ldots\ldots\ldots \quad (1)$$

the residuals are

$$x_1 - a, \quad x_2 - a, \quad \ldots \quad x_n - a,$$

and their sum is

$$\Sigma x - na,$$

which is zero by equation (1).

When the observations are represented by points, as in the preceding article, the geometrical mean point or *centre of gravity* of these points is the point whose abscissa is $a$, and, when this point is taken as the origin, the sum of the positive abscissas of observation points is equal to the sum of the negative abscissas.

*Weights.*

**10.** When the observations are not made under the same circumstances, and are therefore not regarded as equally good, a greater relative importance can be given to a better observation by treating it as equivalent to more than one occurrence of the same observed value in a set of equally good observations. For example, if there were two observations giving the observed values $x_1$ and $x_2$, and the first observation were regarded as the best, we might proceed as if the observed value $x_1$ occurred twice and $x_2$ once in a set of three observations equally good. The arithmetical mean would then be

$$\frac{2x_1 + x_2}{3}.$$

In this process we are said to give to the observations the relative *weights* of 2 and 1. The weight may be regarded as the numerical measure of the influence of the observation upon the arithmetical mean.

**11.** In general, $p_1, p_2, \ldots p_n$ being taken as the weights of the observations $x_1, x_2, \ldots x_n$, the arithmetical mean with these weights is

$$a = \frac{p_1x_1 + p_2x_2 + \ldots + p_nx_n}{p_1 + p_2 + \ldots + p_n} = \frac{\Sigma px}{\Sigma p}.$$

This expression is called the *weighted arithmetical mean.* When the weights are integers, it is the same as the arithmetical mean of $\Sigma p$ observations, of which $p_1$ give the observed value $x_1$, $p_2$ the observed value $x_2$, and so on. But, since only the ratios of the weights affect the result, it is not necessary to suppose them to be integers.

It is easily shown, as in Art. 9, that, if the residuals are multiplied by the weights, the algebraic sum of the results is zero. Again, when as in that article the observations are represented by points, the point whose abscissa is the weighted mean is the centre of gravity of bodies placed at the observation points having weights proportional to $p_1, p_2, \ldots p_n$.

**12.** The weight of a result obtained by the rule given above is defined to be the sum of the weights of its constituents; so that, because

$$a\Sigma p = \Sigma px,$$

the product of a result by its weight is equal to the sum of the like products for its constituents. It follows that, in obtaining the final result, we may for any group of observations substitute their mean with the proper weight.

In the case of observations supposed equally good, the weight of each is taken equal to unity, and then the weight of the mean is the number of observations.

### *The Probable Value.*

**13.** The *most probable value* of the observed quantity, or simply the *probable value*, in the ordinary sense of the expression signifies that which, in our actual state of knowledge, we are justified in considering as more likely than any other to be the true value. In this sense, the arithmetical mean is the most

probable value which can be derived from observations considered equally good. This is, in fact, equivalent to saying that we accept the arithmetical mean as the best rule for combining the observations, having no reason either theoretical or practical for preferring any other.*

But, if instead of a rule of combination we adopt a theory with respect to the nature of accidental errors, the probable value will depend upon the adopted theory. To become the subject of mathematical treatment such a theory must take the shape of a law of the probability of accidental errors, as will be explained in a subsequent section. Since, in the nature of things, this law can never be absolutely known, and since moreover it probably differs with differing circumstances of observation, the most probable value in this technical sense is itself unknown. But when the expression is used without specifying the law of probability, it signifies the value which is the most probable in accordance with the generally accepted law of probability. Before proceeding to this law, we shall consider, in the following section, the principles of probability so far as we shall need to apply them.

*Examples.*

1. Show that the formula $nf(a) = \Sigma f(x)$ determines a mean value of $n$ quantities for any form of the function $f$, and that the geometric mean is included in this rule.

2. Except when $f(x) = cx$ in Ex. 1, the position of the point whose abscissa is $a$ is dependent upon the position of the origin as well as upon the observation points.

* That the most probable value, when there are but two observations, is their arithmetical mean follows rigorously from the hypothesis that positive and negative errors are equally probable. The property of the arithmetical mean pointed out in Art. 12 shows that the result for three observations is expressible as a function of the result for two of them and the third observation, and so on for four or more observations. It was upon the assumption that the most probable value must possess this property that Encke based his so-called proof that the arithmetical mean is the most probable value for any number of observations (*Berliner Astronomisches Jahrbuch* for 1834, pp. 260–262).

3. If the values of $x$ are nearly equal in Ex. 1, the result of the formula is nearly equivalent to a weighted arithmetical mean in which the weights are proportional to $f'(\frac{1}{2}x_1+\frac{1}{2}a)$, $f'(\frac{1}{2}x_2+\frac{1}{2}a)$, etc.

4. When a mean value is determined by an equation of the form $\Sigma f(x-a)=0$, the position of the point whose abscissa is $a$ is independent of the origin. Give the cubic determining $a$ when $\Sigma(x-a)^3=0$, and show that one root only is real.

5. Prove that the weighted arithmetical mean of values of $x+y$ is the sum of the like means of the values of $x$ and of the values of $y$ respectively.

## III.

### Principles of Probability.

*The Measure of Probability.*

14. The *probability* of a future event is the measure of our reasonable expectation of the event in our present state of knowledge of its causes. Thus, not knowing any reason to the contrary, when a die is to be thrown we assign an equal probability to the several events of the turning up of its six different faces. We say, therefore, that the probability or chance that the ace will turn up is 1 to 5, or better, 1 out of 6, hence the fraction $\frac{1}{6}$ is taken as the measure of the probability. Thus the probability of an event which is one of a set of equally likely events, one of which must happen, is the fraction whose numerator is unity and whose denominator is the number of these events. Obviously, *the probability of an event which can happen in several ways is the sum of the probabilities of the several ways.* Thus if the die had two blank faces, the probability that one of them would turn up would be $\frac{2}{6}$ or $\frac{1}{3}$. The sum of the probabilities of all the possible events is unity, which represents the certainty that some one of the events will happen.

*Compound Events.*

15. An event which consists of the joint occurrence of two independent events is called a *compound event.* By independent events we mean events such that the occurrence or non-occurrence of the first has no influence upon the occurrence or non-occurrence of the second. For example, the throwing of sixes with a pair of dice is a compound event consisting of the turning up of a special face of each die. The whole number of compound events is evidently the product of the numbers of simple events; and, since the several probabilities are the reciprocals

of these numbers, the probability of the compound event is the product of the probabilities of the simple events. Thus, when a pair of dice is thrown we have $6 \times 6 = 36$ compound events, and the probability of a special one, such as the throwing of sixes, is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

In like manner, if more than two simple events are concerned, it is easily seen that, in general, *the probability of a compound event is the product of the probabilities of the independent simple events of whose joint occurrence it consists.*

**16.** A compound event may happen in different ways, and then, of course, the probabilities of these independent ways must be added. For example, six and five may be thrown in two ways, that is to say, two of the 36 equally likely events consist of the combination six and five, hence the chance is $\frac{2}{36}$ or $\frac{1}{18}$. A throw whose sum amounts to 10 can occur in three ways, therefore its chance is $\frac{3}{36}$ or $\frac{1}{12}$.

### *Repeated Trials.*

**17.** When repeated opportunities for the occurrence or non-occurrence of the same set of events can be made to take place under exactly the same circumstances, equally probable events will tend to occur with the same frequency. Therefore, in a large number of such opportunities or *trials*, the relative frequency of the occurrence of an event which can happen in $m$ ways and fail in $n$ ways (the $m + n$ ways of both kinds corresponding to $m + n$ equally probable elementary events) will tend to the value $\frac{m}{m+n}$, which is the fraction expressing the probability of the event. This is commonly expressed by saying that the ratio of the number of occurrences of an event to the whole number of trials will "*in the long run*" be the fraction which expresses the probability. The correspondence of this frequency in the long run with the estimated probability forms the only mode, though an uncertain one, of submitting our results to the test of experience.

*The Probability of Values belonging to a Continuous Series.*

**18.** In the examples given in the preceding articles, the equally probable elementary events, which are the basis of our estimate of probability, form a limited number of distinct events, such as the turning up of the different faces of a die. But, in many applications, these events belong to a consecutive series, incapable of numeration. For example, suppose we are concerned with the value of a quantity $x$, of which it is known that any value between certain limits $a$ and $b$ is possible; or, what is the same thing, the position of the point $P$, whose abscissa is $x$, when $P$ may have any position between certain extreme points $A$ and $B$. We cannot now assign any finite measure to the probability that $x$ shall have a definite value, or that $P$ shall fall at a definite point, because the number of points upon the line $AB$ is unlimited. We have rather to consider the probability that $P$ shall fall upon a definite segment of the line, or that the value of $x$ shall lie between certain limits.

**19.** It is customary, however, to compare the probabilities that $P$ shall fall at certain points. Suppose in the first place

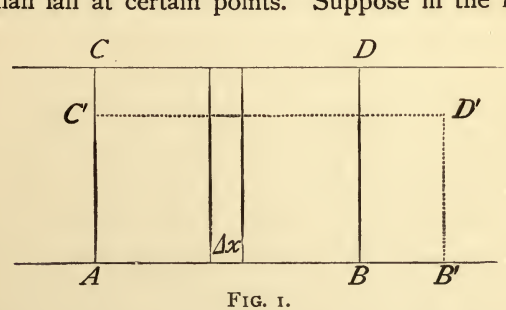


FIG. 1.

that, *when any equal segments of the line AB are taken, the probabilities that P shall fall in these segments are equal.* In this case, *the probability that P shall fall at a given point is said to be constant for all points of the line.* Let $\Delta x$ be a segment of the line $AB$; then, if the probability for all points of $AB$ is constant, it readily follows from the definition just given that the

probability that $P$ shall fall in the segment $\Delta x$ is proportional to $\Delta x$. Since we suppose it certain that $P$ shall fall somewhere between $A$ and $B$, this probability will be represented by

$$\frac{\Delta x}{AB} \quad \text{or} \quad \frac{\Delta x}{b-a}.$$

Let an ordinate $y$ be taken such that $y\Delta x$ is the value of this probability; then

$$y = \frac{1}{b-a},$$

and, constructing as in Fig. 1 the line $CD$ having this constant ordinate, the probabilities for any segments of $AB$ are the corresponding rectangles contained between the axis and the line $CD$ For different values of the limiting space $AB$ in which $P$ may fall, $y$ varies in inverse ratio. Thus, if $AB$ is changed to $AB'$, the new ordinate $AC'$ or $y'$ is such that $y'.AB' = y.AB$, each of the areas $ACDB$ and $AC'D'B'$ being equal to unity. The two values of $y$ are said to determine the relative probabilities that $P$ shall fall at a given point in the two cases.

### *Curves of Probability.*

**20.** Taking now the case in which the probability is not constant for all points, let $AB$ be divided into segments, and let rectangles be erected upon them, the area of each rectangle representing the probability that $P$ shall fall in the corresponding segment. The heights of these rectangles will now differ for the different segments. Denoting the height for a given segment $\Delta x$ by $y$, the relative values of $y$ for any two segments determine, as explained in the preceding article, the relative probability that $P$ shall fall at a given point in one or the other of the segments, on the hypothesis that the probability is constant throughout the segment. They may thus be said to measure the *mean values* of the probabilities for given points taken in the various segments. The sum of the areas of the rectangles will, of course be unity; that is, $\Sigma y\Delta x = 1$.

**21.** If we now subdivide the segments, the figure composed

of the sum of the rectangles will approach more and more nearly, as we diminish the segments without limit, to a curvilinear area, and the variable ordinate of the limiting curve will measure the continuously varying probability that $P$ shall fall at a given point of the line $AB$.

The value of $y$ is now a continuous function of $x$ the abscissa of the corresponding point, and, putting $y = f(x)$, the function $f(x)$ is said to express the *law of the probability of the value* $x$.

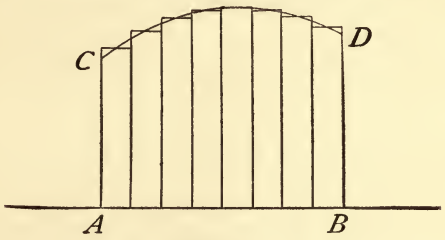


FIG. 2.

The curve $y = f(x)$ is the *probability curve* corresponding to the given law $f(x)$. The entire area $ACDB$, Fig. 2, whose value is $\int_a^b y\,dx$ (which is the limit of $\Sigma y \Delta x$; see Int. Calc., Art. 99), $a$ and $b$ being the limiting values between which $x$ certainly falls, is equal to unity. In general, for any limits the value of the integral $\int_\alpha^\beta y\,dx$ is the probability that $x$ falls between the values $\alpha$ and $\beta$. The element $y\,dx$ of this integral may be called the *element of probability* for the value $x$. It is sometimes called the probability that the value shall fall between $x$ and $x + dx$, it being in that case understood that $dx$ is taken so small that the probability may be regarded as constant in this interval.

**22.** As an illustration of what precedes, suppose it to be known that the value of $x$ must fall between zero and $a$, and that the probabilities of values between these limits are proportional to the values themselves. These conditions give

$$y = cx,$$

and

$$\int_0^a y\,dx = 1,$$

whence, substituting and integrating,

$$\frac{ca^2}{2} = 1, \quad \text{or} \quad c = \frac{2}{a^2}.$$

Hence the law of probability in this case is

$$y = \frac{2x}{a^2}.$$

We may now find the probability that $x$ shall fall between any given limits. For example, the probability that $x$ shall exceed $\frac{1}{2}a$ is represented by

$$P = \int_{\frac{1}{2}a}^{a} y dx = \frac{2}{a^2} \int_{\frac{1}{2}a}^{a} x dx = \frac{3}{4}.$$

Thus the odds are 3 to 1 that $x$ exceeds $\frac{1}{2}a$ when the law of probability is that proposed.

*Mean Values under a given Law of Probability.*

**23.** When a quantity $x$ has a given law of probability, we have frequently occasion to consider what would be its *mean* or *average value* "in the long run," that is to say, the arithmetical mean of its values, supposing them to occur in a large number of trials with the frequency indicated by the given law of probability. See Art. 17.

Let us suppose, in the first place, that only a limited number of distinct values, say

$$x_1, x_2, \ldots x_m,$$

are possible. Let $P_1, P_2 \ldots P_m$ be the proper fractions which represent the respective probabilities of these values. Then, in a large number $n$ of trials, the number of times in which the distinct values $x_1, x_2 \ldots x_m$ occur will be

$$nP_1, \; nP_2, \ldots nP_m$$

respectively. The arithmetical mean mentioned above is, therefore,

$$\frac{nP_1x_1 + nP_2x_2 + \ldots + nP_mx_m}{n(P_1 + P_2 + \ldots P_m)};$$

that is, $$P_1x_1 + P_2x_2 + \ldots + P_mx_m,$$

or $$\Sigma Px.$$

That is to say, the mean value is found by multiplying the $m$ distinct values by their probabilities and adding the results.*

**24.** Next, supposing a continuous series of values possible, let $y\Delta x$ be taken, as in Art. 20, to represent the probability that $x$ falls between $x$ and $x + \Delta x$. Evidently, in each term of $\Sigma Px$, we must now substitute this expression for $P$, and for $x$ some intermediate value between $x$ and $x + \Delta x$. When we pass to the limit, in which $y$ becomes a continuous function of $x$, this sum becomes

$$\int_a^b xydx,$$

which is thus the mean value of $x$, when $y$ is the function expressing its law of probability and $a$ and $b$ its extreme possible values.

For example, with the law of probability considered in Art. 22, namely,

$$y = \frac{2x}{a^2},$$

the mean value of $x$ is

$$\frac{2}{a^2}\int_0^a x^2dx = \frac{2}{3}a.$$

**25.** In the same manner it may be shown that, if $y = f(x)$ expresses the law of probability of $x$, the mean value of any function $F(x)$ is

$$\int_a^b F(x)f(x)dx.$$

* The "value of an expectation" is an instance of a mean value. Thus, if $x_1$ is the value to be received in case a certain event whose probability is $P_1$ happens, $x_2$ the value to be received if an event whose probability is $P_2$ happens, and so on for $m$ distinct events, one of which must happen, then the mean value $\Sigma Px$ is called the value of the expectation.

Thus, again taking the law of probability $y = \frac{2x}{a^2}$, the mean value of $x^2$* is

$$\frac{2}{a^2}\int_0^a x^3 dx = \frac{a^2}{2}.$$

Again, that of $\frac{1}{x}$ is

$$\frac{2}{a^2}\int_0^a dx = \frac{2}{a}.$$

**26.** If all values between $a$ and $b$ are equally probable, the element of probability is $\frac{dx}{b-a}$; thus the mean value of $x$, in this case, is

$$\int_a^b \frac{x dx}{b-a} = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2},$$

which is the same as the arithmetical mean between the limiting values. Again, the mean value of $x^2$, in this case, is

$$\int_a^b \frac{x^2 dx}{b-a} = \frac{b^3 - a^3}{3(b-a)} = \tfrac{1}{3}(b^2 + ab + a^2).$$

*The Probability of Unknown Hypotheses.*

**27.** No distinction can be drawn between the probability of an uncertain future event and that of an unknown contingency, in a case where the decisive "event" has indeed happened, but we remain in doubt with regard to it because only probable evidence

* It should be noticed that if $z = F(x)$, the law of probability for $z$ is not found by simply expressing $f(x)$ as a function of $z$. It is necessary to transform the element of probability $f(x)dx$, which expresses the probability that $x$ falls between $x$ and $x + dx$, and therefore represents also the probability that $z$ falls between $z$ and $z + dz$. Thus, in the present case, putting $z = x^2$,

$$f(x)dx = \frac{2x}{a^2}dx = \frac{dz}{a^2},$$

which indicates that all values of $z$ between 0 and $a^2$ are equally probable when, as supposed in Art. 22, the probability of a value of $x$ is proportional to the value itself.

is known to us. In any case, the probability is a mental estimate of *credibility* depending only upon the known data, and therefore subject to change whenever new evidence becomes known. Let there be two hypotheses $A$ and $B$, one of which must be true, and which so far as we know are equally probable, and suppose that a trial is to be made which on either hypothesis may eventuate in one or the other of two ways; in other words, that an event $X$ may or may not happen. Suppose, further, that on the hypothesis $A$ the probability of $X$ is $a$, and on the hypothesis $B$ the probability of $X$ is $b$. Now it is clear that after the trial has been made and the event $X$ has happened, we are entitled to make a different estimate of the relative credibilities of the hypotheses $A$ and $B$.

**28.** To obtain the new measures of the probabilities of $A$ and $B$, we employ the notion of relative frequency in the long run. Let us then consider a great number of cases of the four kinds which before the event $X$ we regard as possible, the frequencies of the different kinds being proportional to their probabilities as we estimate them before the event. The hypotheses $A$ and $B$ respectively are true in an equal number of cases, say $n$, of each. The event $X$ will happen in $na$ of the cases in which $A$ is true, and not happen in $n(1-a)$ cases. Again, $X$ will happen in $nb$ cases in which $B$ is the true hypothesis, and not happen in $n(1-b)$ cases.

Now, since $X$ has actually happened, from the whole number, $2n$, of cases we must exclude those in which $X$ does not happen, and consider only the $na + nb$ cases in which $X$ does happen.

Attending only to these cases, the relative frequency of those in which $A$ and $B$ respectively are true is the measure of our present estimate of their relative probability. Hence these probabilities are in the ratio $a:b$, that is, the probability of $A$ is $\frac{a}{a+b}$, and that of $B$ is $\frac{b}{a+b}$.

**29.** As an illustration, suppose there are two bags, $A$ and $B$, containing white and black balls, $A$ containing 3 white and 5

black balls, $B$ containing 5 white and 1 black ball. One of the bags is chosen at random, and then a ball is drawn at random from the bag chosen. The ball is found to be white; what is the probability that the bag $A$ was chosen? Here $a = \frac{3}{8}$, since three out of eight balls in $A$ are white, and $b = \frac{5}{6}$; hence the probabilities are in the ratio $\frac{3}{8} : \frac{5}{6}$ or 9 : 20. The probability that the bag was $A$ is therefore $\frac{9}{29}$.

Again, suppose $A$ is known to contain only white balls, and $B$ an equal number of white and black. If a white ball is drawn $a = 1$, $b = \frac{1}{2}$, the odds in favor of $A$ are 2 : 1 or the probability of $A$ is $\frac{2}{3}$. But if a black ball had been drawn, we should have had $a = 0$, $b = \frac{1}{2}$, the probability of $A$ is zero, that is, it is certain that the bag chosen was not $A$.

**30.** If there are other hypotheses besides $A$ and $B$ consistent with the event $X$, the same reasoning as in Art. 28 establishes the general theorem that *the probabilities of the several hypotheses, which before an event $X$ were considered equally probable,* are after the event proportional to the numbers which before the event express the probabilities of $X$ on the several hypotheses.*

The various hypotheses in question may consist in attributing different values to an unknown quantity $x$, and these values may constitute a continuous series. The probabilities of the various values will then be proportional to the corresponding probabilities of the event $X$. Hence, to find the law of the probability of $x$, it is only necessary to determine a constant in the same manner that $c$ is determined in Art. 22.

In particular it is to be noticed that, *of all the values of an unknown quantity which before the occurrence of a certain event were equally probable, that one is after the event the most probable which before the event assigned to it the greatest probability.*

* If this is not the case, the probabilities before the event are called the antecedent or *a priori* probabilities, and the theorem is that the ratio of the antecedent probabilities is to be multiplied by the probabilities of $X$ on the several hypotheses, in order to find the ratio of the probabilities after the event.

*Examples.*

1. From $2n$ counters marked with consecutive numbers two are drawn at random; show that the odds against an even sum are $n$ to $n-1$.

2. $A$ and $B$ play chess, $A$ wins on an average 2 out of 3 games; what is the chance that $A$ wins exactly 4 games out of the first six? $\frac{80}{243}$.

3. A domino is chosen from a set and a pair of dice is thrown; what is the chance that the numbers agree? $\frac{1}{28}$.

4. Show that the chance of throwing 9 with two dice is to the chance of throwing 9 with three dice as 24 to 25.

5. $A$ and $B$ shoot alternately at a mark. $A$ hits once in $n$ times, $B$ once in $n-1$ times; show that their chances of first hit are equal, and find the odds in favor of $B$ after $A$ has missed the first shot. $n$ to $n-2$.

6. $A$ and $B$ throw a pair of dice in turn. $A$ wins if he throws numbers whose sum is 6 before $B$ throws numbers whose sum is 7; show that his chance is $\frac{30}{61}$.

7. $A$ walks at a rate known to be between 3 and 4 miles an hour. He starts to walk 20 miles, and $B$ starts one hour later, walking at the rate of 4 miles an hour. What is the chance of overtaking him: 1° if all distances per hour between the limits are equally probable; 2° if all times per mile between the limits are equally probable? 1°, 1 to 2; 2°, 2 to 3.

8. If all values of $x$ between 0 and $a$ are possible and their probabilities are proportional to their squares, show that the probability that $x$ exceeds $\frac{1}{2}a$ is $\frac{7}{8}$, and find the mean value of $x$. $\frac{3}{4}a$.

9. If, in the preceding example, we are informed that $x$ exceeds $\frac{1}{4}a$, how is the probability affected, and what is now the mean value of $x$? $\frac{45}{56}a$.

10. If two points be taken at random upon a straight line $AB$, whose length is $a$, and $X$ denote that which is nearest $A$, show that the curve of probability for $X$ is a straight line passing through $B$, and find the mean value of $AX$. $\frac{1}{3}a$.

11. On a line $AB$, whose length is $a$, a point $Z$ is taken at random, and then a point $X$ is taken at random upon $AZ$. Determine the probability curve for $AX$, or $x$, and the mean value of $x$.

$$y = \frac{1}{a} \log \frac{a}{x}\,; \quad \frac{a}{4}.$$

12. Two points are taken at random on the circumference of a circle whose radius is $a$. Show that the chord is as likely as not to exceed $a\sqrt{2}$, but that the average length of the chord is $\frac{4a}{\pi}$.

13. In a semicircle whose radius is $a$, find the mean ordinate: 1° when all points of the semi-circumference are equally probable; 2° when all points on the diameter are equally probable.

$$1°, \ \frac{2a}{\pi}\,; \quad 2°, \ \frac{\pi a}{4}.$$

14. A card is missing from a pack; 13 cards are drawn at random and found to be black. Show that it is 2 to 1 that the missing card is red.

15. A card has been dropped from a pack; 13 cards are then drawn and found to be 2 spades, 3 clubs, 4 hearts, and 4 diamonds. What are the relative probabilities that the missing card belongs to the suits in the order named? 11 : 10 : 9 : 9.

16. $A$ and $B$ play at chess: when $A$ has the first move the odds are 11 to 6 in favor of $A$, but when $B$ has the first move the odds are only 9 to 5. $A$ has won a game; what are the odds that he had the first move? 154 to 153.

17. The odds are 2 to 1 that a man will write 'rigorous' rather than 'rigourous.' The word has been written, and a letter taken at random from it is found to be 'u'; what are now the odds? 9 to 8.

18. A point $P$ was taken at random upon a line $AB$, and then a point $C$ was taken at random upon $AP$. If we are informed that $C$ is the middle point of $AB$, what is now the probability curve of $AP$?

$$y = \frac{1}{x \log 2}.$$

## IV.

### The Law of Probability of Accidental Errors.

*The Facility of Errors.*

**31.** If observations made upon the same magnitude could be repeated under the same circumstances indefinitely, only a limited number of observed values, which are exact multiples of the least count of the instrument, would occur, and the relative frequency with which they occurred would indicate the law of the probability of the observed values, that is to say, the *law of facility* with which the corresponding errors are committed. In the theory of errors, however, it is necessary to regard all observed values between certain limits as possible, so that when they are laid down upon a line as abscissas, the law of facility may be represented by a continuous curve, as explained in Art. 21. This is in fact equivalent to supposing the least count diminished without limit.

The curve thus obtained is the probability curve for an observed value; and, if the point representing the true value be taken as origin, the abscissas become errors, and the curve becomes the probability curve for accidental errors committed under the given circumstances.

**32.** The probability curves corresponding to different circumstances of observation would differ somewhat, but in any case would present the following general features. In the first place, since errors in defect and in excess* are equally likely to occur, the curve must be symmetrical to the right and left of the point which represents the true value of the observed quantity. In the next place, since accidental errors are made up of elemental errors (Art. 3) which, as they may have either direction, tend

*There is usually no distinction in kind between these: either direction may be taken as positive, and errors of a given magnitude in one direction or the other are equally likely to occur.

to cancel one another, small errors are more frequent than large ones, so that the maximum ordinate occurs at the central point. In the third place, since large errors (which can only result when most of the elemental errors have the same direction and their greatest magnitudes) are rare, and errors beyond some undefined limit do not occur, the curve must rapidly approach the axis of $x$ both to the right and left, so that the ordinate (which can never become negative) practically vanishes at an indefinite distance from the central point.

**33.** If $y = \varphi(x)$ is the equation of the curve referred to the central point as origin, the general features mentioned above are equivalent to the statements: first, that $\varphi(x)$ is an even function, that is, a function of $x^2$; secondly, that $\varphi(0)$ is its maximum value; thirdly, that it is a decreasing function of $x^2$, and practically vanishes when $x$ is large. Since it is impracticable to select the function $\varphi$ in such a manner that $\varphi(x)$ shall be constantly equal to zero when $x$ exceeds a certain limit, the last condition requires that the curve shall have the axis of $x$ for an asymptote; in other words, we must have $\varphi(\pm\infty) = 0$.

When regarded as the curve of probability of an observed value, the equation is $y = \varphi(x - a)$, where $a$ is the true value of the observed quantity, the origin now corresponding to the zero point of the measurements.

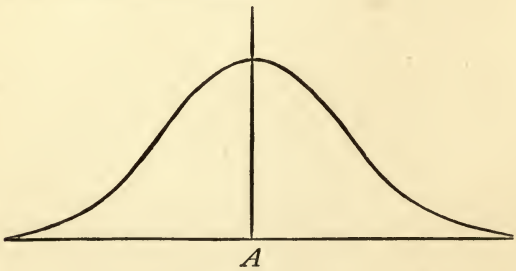


FIG. 3.

The general form of the curve of probability of an observed value will therefore be similar to that given in Fig. 3, in which $A$ is the point whose abscissa $a$ is the true value.

*The Probability of an Error between given Limits.*

**34.** If the law of probability of error for a given observation is

$$y = \varphi(x),$$

the probability that the error of an observation shall lie between $\alpha$ and $\beta$ will, in accordance with Art. 21, be expressed by

$$P = \int_{\alpha}^{\beta} \varphi(x)dx,$$

provided that the value of this integral for the whole range of possible errors is unity. Since we suppose the function $\varphi(x)$ to fulfil the conditions given in Art. 32, we may include all errors in the range of the integral, because the probability of large errors practically vanishes. We therefore write

$$\int_{-\infty}^{\infty} \varphi(x)dx = 1.$$

That is to say, the whole area between the curve and the axis in Fig. 3 is assumed to be unity.

**35.** If $\Delta x$ represents the least count of the instrument, the probability that an observation shall be recorded with the value $x$ will be represented by

$$\int_{x-\frac{1}{2}\Delta x}^{x+\frac{1}{2}\Delta x} \varphi(x)dx.$$

If $\Delta x$ is so small that $\varphi(x)$ may be regarded as constant over the interval, the value of this integral is

$$\varphi(x)\Delta x.$$

The product $\varphi(x)dx$, which is the *element of probability*, being the element of the area which represents the probability, is therefore called the probability of an error between $x$ and $x + dx$, and is sometimes written in the form

$$\int_{x}^{x+dx} \varphi(x)dx.$$

*The Probability of a System of Observed Values.*

**36.** Let $x_1, x_2, \ldots x_n$ be a series of observed values of a quantity whose true value is $a$, the observations being all made under the same circumstances. Then

$$x_1 - a, \quad x_2 - a, \quad \ldots \quad x_n - a$$

are the errors of observation; and,

$$y = \varphi(x - a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

being the law of facility of the errors, the probability before the first observation is made that $x_1$ shall be the first observed value is $\varphi(x_1 - a)\Delta x$, where $\Delta x$ is the least count of the instrument. In like manner, the probability that $x_2$ shall be the second observed value is $\varphi(x_2 - a)\Delta x$, and so on.

It follows, in accordance with the principle explained in Art. 15, that, if $P$ denote the probability of the compound event consisting in the occurrence of the $n$ observed values, then, before the observations were made we should have

$$P = \varphi(x_1 - a)\,\varphi(x_2 - a) \ldots \varphi(x_n - a)\,\Delta x^n. \quad . \quad (2)$$

*The Most Probable Value derivable from a given System of Observed Values.*

**37.** Supposing the form of the function $\varphi$ to be known, the value of $P$ given above is a known function of the unknown true value $a$. Regarding different values of $a$ as hypotheses all equally probable before the observations were made, the principle enunciated in Art. 30 shows that that value of $a$ is most probable which assigns to $P$ the greatest value.

The value of $a$ thus found, or most probable value, depends therefore in part upon the form of the function $\varphi$, this being the mathematical expression of a law which, as stated in Art. 13, can never be absolutely known. We proceed to the method of Gauss, which consists in determining the form of $\varphi$ in accordance with which the arithmetical mean becomes the most probable value.

*The Form of $\varphi$ corresponding to the Arithmetical Mean.*

**38.** If we put

$$\log \varphi(x-a) = \psi(x-a), \quad \ldots \quad (1)$$

we have from equation (2), Art. 36,

$$\log P = \psi(x_1-a)+\psi(x_2-a)+\ldots+\psi(x_n-a)+n\log \Delta x, \quad (2)$$

and $a$ is to be so taken that $P$, and therefore $\log P$, shall be a maximum. Hence, putting $\psi'$ for the derivative of $\psi$, we have by differentiation with respect to $a$,

$$\psi'(x_1-a)+\psi'(x_2-a)+\ldots+\psi'(x_n-a)=0. \quad (3)$$

Denoting the quantities

$$x_1-a, \quad x_2-a, \quad \ldots \quad x_n-a,$$

which are the residuals, by $v_1, v_2, \ldots v_n$, this equation may be written

$$\psi'(v_1)+\psi'(v_2)+\ldots+\psi'(v_n)=0. \quad \ldots \quad (4)$$

Supposing now the value of $a$ which satisfies equation (3) to be the arithmetical mean, we have by Art. 9,

$$v_1+v_2+\ldots+v_n=0. \quad \ldots \quad (5)$$

We wish therefore to find the form of the function $\psi'$ such that equation (4) is satisfied by every set of values of $v_1, v_2, \ldots v_n$ which satisfy equation (5). For this purpose, suppose all the values of $v$ except $v_1$ and $v_2$ to remain unchanged while equation (5) is still satisfied. The new values may then be denoted by $v_1+k$ and $v_2-k$, in which $k$ is arbitrary. Substituting the new values in equation (4), the sum of the first two terms must remain unchanged since all of the other terms are unchanged; therefore,

$$\psi'(v_1+k)+\psi'(v_2-k)=\psi'(v_1)+\psi'(v_2);$$

whence

$$\frac{\psi'(v_1+k)-\psi'(v_1)}{k}=\frac{\psi'(v_2)-\psi'(v_2-k)}{k}. \quad \ldots \quad (6)$$

When $k$ is diminished without limit this becomes

$$\frac{d\psi'(v)}{dv}\Big]_{v_1} = \frac{d\psi'(v)}{dv}\Big]_{v_2};$$

hence, because $v_1$ and $v_2$ are independent, we infer that

$$\frac{d\psi'(v)}{dv} = c, \quad \ldots\ldots\ldots \quad (7)$$

where $c$ is an unknown constant.

The integral of equation (7) is $\psi'(v) = cv + c'$: but, substituting in equation (4), we find $c' = 0$; hence

$$\psi'(v) = cv. \quad \ldots\ldots\ldots \quad (8)$$

Integrating again,

$$\psi(v) = \tfrac{1}{2}cv^2 + c'',$$

or, by equation (1),

$$\log \varphi(v) = -h^2v^2 + c'', \quad \ldots\ldots \quad (9)$$

in which we have written $-h^2$ for the constant $\frac{1}{2}c$, because we know from Art. 33 that $\varphi(v)$ is a decreasing function of $v^2$.

Finally, equation (9) gives

$$\varphi(v) = Ce^{-h^2v^2}, \quad \ldots\ldots\ldots \quad (10)$$

which is accordingly the law of facility of error which makes the arithmetical mean the most probable value.

### *The Determination of the Value of C.*

**39.** The constants $C$ and $h$ which arise in the above process are not independent; for, $x$ denoting the error as in Art. 34, we must have

$$\int_{-\infty}^{\infty} \varphi(x)dx = 1.$$

Substituting from equation (10) above, this gives

$$C\int_{-\infty}^{\infty} e^{-h^2x^2}dx = 1, \quad \ldots\ldots\ldots \quad (11)$$

by which the value of $C$ in terms of $h$ may be found.

A convenient mode of evaluating the definite integral involved in this equation results from the consideration of the solid included between the plane of $xy$ and the surface generated by the revolution of the curve

$$z = e^{-h^2x^2}$$

about the axis of $z$. Using polar coordinates in the plane of $xy$, the equation of the surface is

$$z = e^{-h^2r^2} = e^{-h^2(x^2+y^2)}. \quad . \quad . \quad . \quad . \quad (2)$$

The volume of the solid in question is therefore expressed by either of the two formulae

$$V = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-h^2x^2} e^{-h^2y^2} dxdy, \quad . \quad . \quad . \quad . \quad (3)$$

and

$$V = \int_0^{2\pi}\int_0^{\infty} e^{-h^2r^2} r dr d\theta. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The second expression is readily evaluated and gives

$$V = \frac{\pi}{h^2} e^{-h^2r^2}\Big]_{\infty}^{0} = \frac{\pi}{h^2}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In equation (3), the limits of integration are independent; hence

$$V = \int_{-\infty}^{\infty} e^{-h^2x^2} dx \cdot \int_{-\infty}^{\infty} e^{-h^2y^2} dy = \left[\int_{-\infty}^{\infty} e^{-h^2x^2} dx\right]^2. \quad . \quad (6)$$

Comparing equations (5) and (6), we have

$$\int_{-\infty}^{\infty} e^{-h^2x^2} dx = \frac{\sqrt{\pi}}{h}.* \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Substituting in equation (1), we have $C = \frac{h}{\sqrt{\pi}}$, and the law of facility becomes

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

* It is readily shown that $\int_{-\infty}^{\infty} e^{-t^2} dt = \Gamma(\frac{1}{2})$, the value of which is $\sqrt{\pi}$: equation (7) may also be derived by putting $t = hx$ in this result.

a law which, it is readily seen, fulfils the conditions given in Art. 32.

**40.** The law of facility expressed in the equation derived above is that which is universally adopted ; in other words, it is assumed that under any circumstances of observation the law of facility will be satisfactorily represented by equation (8) if the value of $h$ be properly determined. The mode of determining the most probable value of $h$ for a given set of observations will be given in the following section.

We proceed to develop the consequences of this law. Among them will, of course, be found the rule of the Arithmetical Mean in accordance with which the law has been derived (see Art. 42). Certain confirmations of the law, both of a theoretic and a practical nature, will also be noticed as they present themselves.

### *The Principle of Least Squares.*

**41.** Substituting the expression now obtained for the function $\varphi$, the expression for the probability of the occurrence of the actual observed values (as estimated before the observations were made, see Art. 36) becomes

$$P = \frac{h^n}{\pi^{\frac{1}{2}n}} e^{-h^2[(x_1-a)^2+(x_2-a)^2+\ldots+(x_n-a)^2]} \Delta x^n. \quad . \quad (1)$$

This expression, regarded as a function of $a$, is obviously a maximum when

$$(x_1 - a)^2 + (x_2 - a)^2 + \ldots + (x_n - a)^2 = \text{a minimum.} \quad (2)$$

Hence the most probable value of the observed quantity $a$, in the case of observations supposed equally good, is *that which assigns the least possible value to the sum of the squares of the residual errors.* This is the statement in its simplest form of *the principle of Least Squares.*

**42.** The rule of the Arithmetical Mean follows directly from the principle of Least Squares. Thus, by differentiation with respect to $a$, we derive from equation (2)

$$x_1 - a + x_2 - a + \ldots + x_n - a = 0;$$

that is, the algebraic sum of the residuals is zero, or

$$a = \frac{\Sigma x}{n};$$

in other words, the arithmetical mean is to be taken as the most probable value.

43. Conversely, we may show directly that the arithmetical mean makes the sum of the squares of the residuals a minimum. For, if $a$ is the arithmetical mean, the residuals are

$$v_1 = x_1 - a, \quad v_2 = x_2 - a, \quad \ldots \quad v_n = x_n - a,$$

and $\Sigma v = 0$. Now if $\delta$ is the error of the arithmetical mean, the true value of the observed quantity is $a - \delta$, and the true expressions for the errors of the observed values are

$$x_1 - a + \delta = v_1 + \delta, \quad \ldots \quad x_n - a + \delta = v_n + \delta.$$

The sum of the squares of the $n$ errors is therefore

$$\begin{aligned} \Sigma(v + \delta)^2 &= \Sigma v^2 + 2\delta\Sigma v + n\delta^2 \\ &= \Sigma v^2 + n\delta^2, \end{aligned}$$

since $\Sigma v = 0$. The minimum value of this expression is obviously $\Sigma v^2$, the value assumed when $\delta = 0$; that is to say, the sum of the squares of the residuals is least when the arithmetical mean is taken as the value of the observed quantity.

### *The Probability Integral.*

44. Taking now the probability curve to be

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}, \quad \ldots \quad (1)$$

the probability of an error between $\alpha$ and $\beta$ in magnitude is

$$\frac{h}{\sqrt{\pi}} \int_\alpha^\beta e^{-h^2x^2} dx,$$

and, in particular, the probability of an error numerically less than $\delta$ is

$$P = \frac{h}{\sqrt{\pi}} \int_{-\delta}^{\delta} e^{-h^2x^2} dx. \quad \ldots \quad (2)$$

If we put $hx = t$, this may be written in the form

$$P = \frac{1}{\sqrt{\pi}} \int_{-h\delta}^{h\delta} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_{0}^{h\delta} e^{-t^2} dt, \quad . \quad . \quad . \quad (3)$$

which shows that $P$ depends solely upon the value of $h\delta$, that is, upon the limiting value of $t$.

Table I gives the values of this integral for values of $t$ from 0 to 2 at intervals of .01. The halves of the tabular numbers are the values of the probability of an error whose reduced value falls between the limits 0 and $t$, and by combining these we can readily find the values of the probability for any given limits.

### *The Measure of Precision.*

**45.** The value of $h$ in the probability curve depends upon the circumstances of observation. Let $h_1$ and $h_2$ be the values of $h$ corresponding to two sets of observations for which the curves

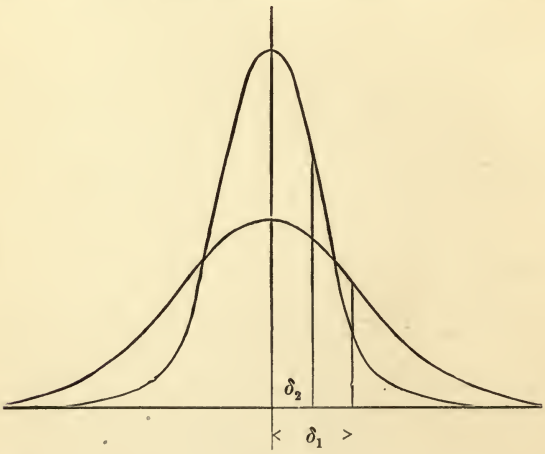


FIG. 4.

are drawn in Fig. 4. The ordinates corresponding to $x = 0$ in the two curves are proportional to the values of $h$. Hence,

because small errors are relatively more frequent in the better set of observations, the value of $h$ for this set will be the larger.

**46.** Let $\delta_1$ be any error, and put

$$h_1\delta_1 = t = h_2\delta_2;$$

then, because $\delta_1$ in the first set of observations and $\delta_2$ in the second set correspond to the same value of $t$ in the probability integral, equation (3), Art. 44, the probability that an error shall be less than $\delta_1$ in the first set is the same as the probability that an error shall be less than $\delta_2$ in the second set. In Fig. 4, for example, we have taken $h_2 = 2h_1$; it follows that $\delta_2 = \frac{1}{2}\delta_1$; that is to say, the probability that an error shall not exceed a given limit in the first case is the same as the probability that an error shall not exceed one half of the given limit in the second case.* The ordinates corresponding to $\delta_1$ and $\delta_2$ in the two curves are drawn in Fig. 4. The areas cut off in the two cases are equal. It is, in fact, readily seen that the second curve might have been derived from the first by reducing the abscissa of each point of the curve to one half its value and at the same time doubling the corresponding ordinate, a process which evidently would not affect the total area, which, as we have seen, must always be equal to unity.

**47.** The ratio of $\delta_1$ and $\delta_2$ which correspond to the same probability may be said to measure *the relative risk of error* in the two cases. Thus, in the example illustrated in Fig. 4, the risk of error in the first case is double that in the second case. It is natural to regard the precision of the observations in the second case as double that of the observations in the first case. So also, in general, the ratio of *precision* is inversely that of the *risk of error;* that is to say, it is the direct ratio of the values of $h$, which are inversely proportional to the corresponding values of $\delta$. Accordingly $h$ is taken as the *measure of precision.*

* This is frequently inaccurately expressed by the statement that the probability of a given error in the first case is the same as that of the half error in the second case.

If the errors in any system of observations are multiplied by the proper values of $h$, the results are the corresponding values of $t$. Errors belonging to different systems may thus be reduced to the same scale, and the values of $t$, or *reduced errors*, will then admit of direct comparison.

### *The Probable Error.*

48. The error which is just as likely to be exceeded as not is called *the probable error.** In other words, the probable error is the value of $\delta$ for which $P = \frac{1}{2}$ in equation (2), Art. 44. Denoting by $\rho$ the corresponding value of $t$ in equation (3) of the same article, we have

$$\frac{1}{2} = \frac{2}{\sqrt{\pi}} \int_0^{\rho} e^{-t^2} dt.$$

The solution of this equation has been found to be

$$\rho = 0.476936,$$

which is the value of $t$ corresponding to the interpolated value $P = 0.5$ in Table I.

Denoting the probable error by $r$, we have then

$$rh = \rho,$$

$$r = \frac{\rho}{h} = \frac{0.4769}{h}.$$

### *The Mean Absolute Error.*

49. The mean value of all possible errors, having regard to their probability or frequency in the long run, is, in accordance with Art. 24,

$$\frac{h}{\sqrt{\pi}} \int_{-\infty}^{\infty} x e^{-h^2x^2} dx.$$

* The "probable error" is thus not the *most* probable error, which is, of course, the error zero, for which the ordinate of the probability curve is a maximum.

The value of this is of course zero, the parts of the integral corresponding to positive and negative errors being equal and having contrary signs. The value obtained by taking both parts of the integral as positive is the mean of the errors taken all positively, or *the mean of the absolute values of the errors.* Denoting this mean by $\eta$, we have

$$\eta = \frac{2h}{\sqrt{\pi}} \int_0^\infty x e^{-h^2x^2} dx;$$

whence

$$\eta = \frac{1}{h\sqrt{\pi}}.$$

*The Mean Error.*

**50.** The mean of all values of the square of the error, having regard to their probabilities, is, in like manner (see Art. 25),

$$\frac{h}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-h^2x^2} dx.$$

The error whose square has this mean value is denoted by $\varepsilon$. On account of its importance in the theory, this error is called the *mean error.* Thus

$$\varepsilon^2 = \frac{h}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-h^2x^2} dx.$$

The value of the definite integral involved in this expression may be deduced from the result found in Art. 39, equation (7), namely,

$$\int_{-\infty}^{\infty} e^{-h^2x^2} dx = \frac{\sqrt{\pi}}{h}.$$

Differentiating with respect to $h$, we have

$$-2h \int_{-\infty}^{\infty} x^2 e^{-h^2x^2} dx = -\frac{\sqrt{\pi}}{h^2},$$

and, substituting in the value of $\varepsilon^2$, we find

$$\varepsilon^2 = \frac{1}{2h^2}, \quad \text{or} \quad \varepsilon = \frac{1}{h\sqrt{2}}.$$

*Measures of the Risk of Error.*

51. We have seen in Art. 47 that the errors corresponding in two different systems to the same value of the reduced error $t$ measure by their ratio the comparative risk of error in the two systems. Thus the error corresponding to any fixed value of $t$ might be taken as the measure of this risk. Accordingly either of the errors

$$r, \quad \eta, \quad \varepsilon,$$

which correspond respectively to the reduced errors

$$\rho, \quad \frac{1}{\sqrt{\pi}}, \quad \frac{1}{\sqrt{2}},$$

may be taken as *the measure of the risk of error** or *inverse measure of precision.*

The probable error $r$ is that which is most frequently employed in practice. Each of the others bears a fixed ratio to $r$, their values being respectively

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* The error $\eta$ was called by De Morgan the *mean risk of error*, because it is the mean expectation of error, using the term expectation in the same sense as in the expression "value of an expectation." (See footnote on page 15.) It corresponds to what is generally called in annuity tables "the expectation of life" for persons of a given age, which should rather be called "the mean duration of survival" for persons of the given age. On the other hand, the probable error $r$ is analogous to the remaining term for which a person of the given age is as likely as not to live. This might be called "the probable term of survival," and its value may differ materially from the mean duration. Thus, according to the Carlisle mortality table, one half of the whole number of persons thirty years old survive for the term of 36.6 years, but the mean duration of life for such persons, as computed from the same table, is only 34.3 years. This indicates that the law of mortality is such that the half which exceed the term of probable survival do so by a total amount less than that by which the other half fall short of it.

In the case of errors the difference falls in the opposite direction. In the long run one half of the errors exceed $r$; and the fact that $\eta > r$ shows that the half which exceed $r$ do so by a total amount greater than that by which the other half fall short of it.

$$\eta = \frac{r}{\rho\sqrt{\pi}} = 1.1829\, r, \quad \ldots\ldots \quad (1)$$

$$\varepsilon = \frac{r}{\rho\sqrt{2}} = 1.4826\, r, \quad \ldots\ldots \quad (2)$$

52. Fig. 5 shows the positions of the ordinates corresponding to $r$, $\eta$ and $\varepsilon$ in the curve of facility of errors

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}.$$

The diagram is constructed for the value $h = 2$.

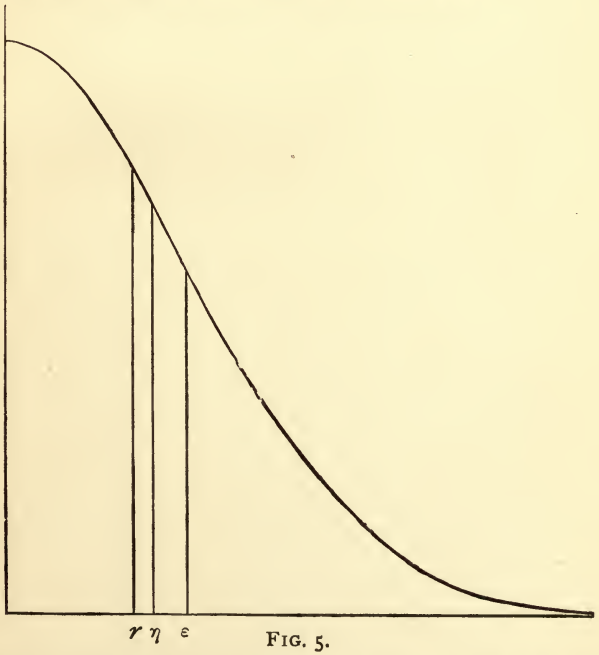


FIG. 5.

From the definitions of the errors it is evident that the ordinate of $r$ bisects the area between the curve and the axes, that of $\eta$ passes through its centre of gravity, and that of $\varepsilon$ passes through its centre of gyration about the axis of $y$.

The advantage of employing in practice a measure of the risk of error, instead of the direct measure of precision, results from the fact that it is of the same nature and expressed in the same units as the observations themselves. It therefore conveys a better idea of the degree of accuracy than is given by the value of the abstract quantity $h$. When the latter is given, it is of course necessary also to know the unit used in expressing the errors.

*Tables of the Probability Integral.*

**53.** The integral $\int_0^t e^{-t^2} dt$ is known as *the error function* and is denoted by Erf $t$.* Table I, which has already been described, Art. 44, gives the values of $\frac{2}{\sqrt{\pi}}$ Erf $t$, which is the probability that an error shall be numerically less than the error $x$, of which the reduced value is $t$. The argument of this table is the reduced error $t$.

But it is convenient to have the values of the probability given also for values of the ratio of the error $x$ to the probable error. Putting $z$ for this ratio, we have, since $hx = t$ and $hr = \rho$,

$$z = \frac{x}{r} = \frac{t}{\rho}.$$

Table II gives, to the argument $z$, the same function of $t$ which is given in Table I; that is to say, the function of $z$ tabulated is

$$P_z = \frac{2}{\sqrt{\pi}} \text{Erf } \rho z,$$

* The integral $\int_t^\infty e^{-t^2} dt$ is denoted by Erfc $t$, being the complement of the error function, so that

$$\text{Erf } t + \text{Erfc } t = \int_0^\infty e^{-t^2} dt = \tfrac{1}{2}\sqrt{\pi}.$$

These functions occur in several branches of Applied Mathematics. A table of values of Erfc $t$ to eight places of decimals was computed by Kramp ("Analyse des Réfractions Astronomiques et Terrestres," Strasbourg, 1799), and from this the existing tables of the Probability Integral have been derived.

which is the probability that an error shall be numerically less than the error $x$ whose ratio to the probable error is $z$.

54. By means of the tables of the probability integral, comparisons have been made between the actual frequency with which given errors occur in a system containing a large number of observations and their probabilities in accordance with the law of facility.

The following example is given by Bessel in the *Fundamenta Astronomiae*. From 470 observations made by Bradley on the right ascensions of Procyon and Altair, the probable error of a single observation was found (by the formula given in the next section) to be

$$r = 0''.2637 .$$

With this value of $r$, the probability that an error shall be numerically less than $0''.1$ is found by entering Table II with the argument

$$z = \frac{0''.1}{0''.2637} = 0.3792 ,$$

and the probability that it shall be less than $0''.2$, $0''.3$ and so on, by entering the table with the successive multiples of this quantity. In the annexed table the first column contains the successive values of the limiting error $x$, the second those of $z$,

| $x$ | $z$ | $P$ | Differences. | Theoretical Nos. of Errors. | Actual Nos. of Errors. |
|---|---|---|---|---|---|
| $0''.1$ | 0.379 | 0.2018 | 0.2018 | 94.8 | 94 |
| 0 .2 | 0.758 | 0.3907 | 0.1889 | 88.8 | 88 |
| 0 .3 | 1.138 | 0.5573 | 0.1666 | 78.3 | 78 |
| 0 .4 | 1.517 | 0.6937 | 0.1364 | 64.1 | 58 |
| 0 .5 | 1.896 | 0.7990 | 0.1053 | 49.5 | 51 |
| 0 .6 | 2.275 | 0.8751 | 0.0761 | 35.8 | 36 |
| 0 .7 | 2.654 | 0.9265 | 0.0514 | 24.2 | 26 |
| 0 .8 | 3.034 | 0.9593 | 0.0328 | 15.4 | 14 |
| 0 .9 | 3.413 | 0.9787 | 0.0194 | 9.1 | 10 |
| 1 .0 | 3.792 | 0.9894 | 0.0107 | 5.0 | 7 |
| $\infty$ | $\infty$ | 1.0000 | 0.0106 | 5.0 | 8 |

and the third the corresponding values of the probability of an error less than $x$ as given by Table II. The fourth column contains the successive differences of these, so that each of the numbers contained in it is the probability of an error falling between the corresponding value of $x$ and that which precedes it. The fifth column contains the multiples of these by 470, which are the theoretical numbers of errors to be expected within the intervals, the last number in the column being the number of errors which should exceed 1″.0. Finally, the last column contains the actual numbers of errors which occurred in the corresponding intervals, as given by Bessel. The agreement between the theoretical and actual numbers is remarkably close, and forms a practical confirmation of the adopted law of facility.

*The Distribution of Errors on a Plane Area.*

**55.** The deviations of the bullet marks in target practice from the point aimed at are of the nature of accidental errors. It is usually assumed that the lateral deviations and the vertical deviations are independent of one another, and that each follows the law of facility for linear errors. We proceed to determine the resulting law of the distribution of the shots upon the plane area.

Let the point aimed at be taken as the origin of coordinates, the horizontal deviation of a shot being denoted by $x$ and the vertical deviations by $y$, and let these deviations be assumed to have the same measure of precision. Then the probability of a horizontal deviation between $x$ and $x + dx$ is

$$\frac{h}{\sqrt{\pi}} e^{-h^2x^2} dx;$$

and for each value of $x$ the probability of a vertical deviation between $y$ and $y + dy$ is

$$\frac{h}{\sqrt{\pi}} e^{-h^2y^2} dy.$$

Hence the probability of hitting the elementary rectangular area $dxdy$, which involves the joint occurrence of these deviations, is

$$\frac{h^2}{\pi} e^{-h^2(x^2+y^2)} dxdy;$$

and, since the probability of hitting an elementary area is proportional to the area, if $a$ denote such an area situated at the point $(x, y)$, the probability of hitting it is

$$\frac{h^2}{\pi} e^{-h^2 r^2} a,$$

where $r$ denotes the distance of $a$ from the origin.

Thus the hypothesis of independent vertical and horizontal deviations, each following the usual law of facility and having the same measure of precision, leads to the conclusion that the facility of the resultant deflection depends solely upon its linear amount, $r$, and not at all upon its direction.* This agrees with

* Sir John Herschel's proof of the law of facility of errors (*Edinburgh Review*, July, 1850) rests upon the assumption that it must possess the property which is above shown to belong to the exponential law. He compares accidental errors to the deviations of a stone which is let fall with the intention of hitting a certain mark, and assumes that the deviations in the directions of any two rectangular axes are independent. But, since there is no reason why the resultant deviations should depend upon their direction, this implies that, $f(x^2)$ being the law of facility, we must have

$$f(x^2)f(y^2) = f(x'^2)f(y'^2) = f(x^2+y^2)f(0)$$

where $x'$ and $y'$ denote coordinates referred to a new set of rectangular axes, so that

$$x^2 + y^2 = x'^2 + y'^2.$$

Now the solution of the functional equation

$$f(x^2)f(v^2) = cf(x^2+y^2)$$

is

$$f(x^2) = ce^{kx^2}$$

where $c$ and $k$ are constants.

There is no *à priori* reason why the deviations in $y$ should, as assumed

the usual custom of judging of the accuracy of a shot solely by its distance from the point aimed at.

### *The Surface of Probability.*

**56.** If at every point of the plane of $xy$ we erect a perpendicular $z$, taking

$$z = \frac{h^2}{\pi} e^{-h^2(x^2+y^2)},$$

we shall have a *surface of probability* analogous to the curve of probability in the case of linear errors. Since the probability of hitting the elementary area $dxdy$ is $zdxdy$, the probability of hitting any area is the value of the double integral

$$\iint z dx dy$$

taken over the given area. That is to say, it is the volume of the right cylinder having this area for its base, and having its upper surface in the surface of probability.

The probability surface is a surface of revolution. The solid included between it and the plane of $xy$ is in fact similar to that employed in Art. 39, in evaluating the integral $\int_{-\infty}^{\infty} e^{-h^2x^2} dx$.

### *The Probability of Hitting a Rectangle.*

**57.** The probability of hitting the rectangle included between the horizontal lines $y = y_1$, $y = y_2$ and the vertical lines $x = x_1$, $x = x_2$ is the double integral

$$\frac{h^2}{\pi} \int_{x_1}^{x_2} \int_{y_1}^{y_2} e^{-h^2x^2} e^{-h^2y^2} dy dx,$$

above, occur with the same relative frequency when $x$ has one value as when it has another; but it is noteworthy that, having made this assumption, no other law of facility of linear deviation would produce a law of distribution in area involving only the distance from the centre. On the other hand, no other law of distribution in area depending only upon $r$ (such for example as $e^{-r}$) would make the law of facility for deviations in $y$ independent of the value of $x$.

which, because the limits for each variable are independent of the other, is equivalent to

$$\frac{h}{\sqrt{\pi}}\int_{x_1}^{x_2} e^{-h^2x^2}dx \,.\, \frac{h}{\sqrt{\pi}}\int_{y_1}^{y_2} e^{-h^2y^2}dy;$$

that is, it is the product of the probabilities that $x$ and $y$ respectively shall fall between their given limits. This result is, of course, nothing more than the expression of the hypothesis made in Art. 55.* If $h$ be known, the values of the factors in the expression (2) may be derived from Table I, as explained in Art. 44.

In particular, putting $x_1 = -\delta$, $x_2 = \delta$, $y_1 = -\delta'$, $y_2 = \delta'$, we have for the probability of hitting a rectangle whose centre is at the origin and whose sides are $2\delta$ and $2\delta'$,

$$p = P_\delta P_{\delta'},$$

where $P_\delta$ and $P_{\delta'}$ are tabular results taken from Table I, if $h$ be given, or from Table II if the probable error of the deviations be given.

For example, for the square whose centre is the origin and whose half side is $r_1$, the probable error of the component deviations, the probability of hitting is $\frac{1}{4}$.

Again, to find the side of the centrally situated square which is as likely as not to be hit, and which therefore may be called *the probable square*, we must determine the value of $\delta$ for which $P_\delta = \sqrt{\frac{1}{2}} = 0.7071$. This will be found to correspond to $t = 0.7437$, whence the side of the square is $2\delta$, where

$$\delta = \frac{t}{h} = \frac{0.7437}{h}.$$

* The property of the probability surface corresponding to the assumption that the relative frequency of the deviations in $y$ is independent of the value of $x$ is that any section parallel to the plane of $yz$ may be derived from the central section in that plane by reducing all the values of $z$ in the same ratio. In accordance with the preceding foot-note, this is the only surface of revolution possessing this property.

*The Probability of Hitting a Circle.*

**58.** Putting $a = 2\pi r dr$ in the expression derived in Art. 55, the probability of hitting the elementary annular area between the circumferences whose radii are $r$ and $r + dr$ is found to be

$$dp = 2h^2 e^{-h^2 r^2} r dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence the probability that the distance of a shot from the point aimed at shall fall between $r_1$ and $r_2$ is

$$p = 2h^2 \int_{r_1}^{r_2} e^{-h^2 r^2} r dr = e^{-h^2 r_1^2} - e^{-h^2 r_2^2}. \quad . \quad . \quad (2)$$

Putting the lower limit $r_1$ equal to zero, we have, for the probability of planting a shot within the circle whose radius is $r$,

$$p = 1 - e^{-h^2 r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

a formula in which $h$ is the measure of the accuracy of the marksman.

*The Radius of the Probable Circle.*

**59.** If we denote by $a$ the value of $r$ corresponding to $p = \frac{1}{2}$ in equation (3) of the preceding article, we shall have

$$e^{-h^2 a^2} = \tfrac{1}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whence

$$a = \frac{\sqrt{\log 2}}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then $a$ is the radius of *the probable circle*, that is, the circle within which a shot is as likely as not to fall, or within which in the long run the marksman can plant half his shots. Thus $a$ is analogous to the probable error in the case of linear deviations, and, being inversely proportional to $h$, may be taken as an inverse measure of the skill of the marksman.

Eliminating $h$ from the formula for $p$ by means of equation (1), we obtain

$$p = 1 - \left(\frac{1}{2}\right)^{\frac{r^2}{a^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Denoting by $n$ the whole number of shots, and by $m$ the number of those which miss a circular target of radius $r$, we may, if $n$ and $m$ be sufficiently large, put

$$1 - p = \frac{m}{n}.$$

Supposing $p$ in equation (3) to be thus determined, we derive the formula

$$a = r\sqrt{\frac{\log 2}{\log n - \log m}},$$

in which the ordinary tabular logarithms may be employed.*

*The Most Probable Distance.*

**60.** Equation (1), Art. 58, shows that the probability of hitting the elementary annulus of radius $r$ is proportional to

$$re^{-h^2r^2}.$$

The value of $r$ which makes this function a maximum is found to be identical with $\varepsilon$, the mean error of the linear deviations, namely,

$$\varepsilon = \frac{1}{h\sqrt{2}},$$

which is therefore the most probable distance† at which a shot can fall.

This distance might, like $a$, be taken as the inverse measure of the skill of the marksman.

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* This is Sir John Herschel's formula for the inverse measure of the skill of the marksman. See "Familiar Lectures on Scientific Subjects," p. 498. London and New York, 1867.

† The point at which the probability is a maximum (that is, where the density of the shots in the long run is the greatest) is of course the origin, at which the ordinate $z$ in the probability surface is a maximum. The value of $r$ here determined is that for which the right cylindrical surface included between the plane of $xy$ and the probability surface is a maximum, that is, the annulus which contains the greatest number of shot in the long run.

*Measures of the Accuracy of Shooting.*

**61.** Any quantity inversely proportional to $h$ might be taken as the measure of the marksman's risk of error, or inverse measure of precision. We may employ for this purpose either $a$, the radius of the probable error, $\epsilon$, the most probable distance, $\delta$, the half side of the probable square (Art. 57), or $r_1$, the probable error of a linear deviation.

The most probable value of $h$ derivable from $n$ given shots will be shown in the next section, Art. 73, to be

$$h = \sqrt{\frac{n}{\Sigma r^2}}.$$

Employing this value of $h$ we have

$$a = \frac{\sqrt{\log 2}}{h} = 0.8326 \sqrt{\frac{\Sigma r^2}{n}},$$

$$\epsilon = \frac{1}{h\sqrt{2}} = 0.7071 \sqrt{\frac{\Sigma r^2}{n}} = \sqrt{\frac{\Sigma r^2}{2n}},$$

$$\delta = \frac{0.7437}{h} = 0.7437 \sqrt{\frac{\Sigma r^2}{n}},$$

$$r_1 = \frac{\rho}{h} = 0.4769 \sqrt{\frac{\Sigma r^2}{n}}.$$

*Examples.*

1. Show that the abscissa of the point of inflexion in the probability curve is the mean error.

2. In 1000 observations of the same quantity how many may be expected to differ from the mean value by less than the probable error, by less than the mean absolute error, and by less than the mean error respectively? 500, 575, 683.

3. An astronomer measures an angle 100 times; if, when the unit employed is 1″, the measure of precision is known to be

$h = \frac{1}{5}$, how many errors may be expected to have a numerical value between 2″ and 4″? 31.

4. In 125 observations whose probable error is 2″, how many errors less than 1″ are to be expected? 33.

5. If the probable error is ten times the least count of the instrument, show that about 27 observations out of 1000 will be recorded with the true value, and 21 will exceed it by an amount equal to the probable error.

6. If $h$ is changed to $mh$ ($m > 1$), errors less than a certain error $x_1$ are more probable, and errors greater than $x_1$ are less probable. Find $t_1$ the reduced value of $x_1$.

$$t_1 = hx_1 = \sqrt{\frac{\log m}{m^2 - 1}}.$$

7. Show that the envelop of the probability curve, when $h$ varies, is the hyperbola

$$xy = \frac{1}{\sqrt{(2\pi e)}},$$

the abscissa of the point of contact being the mean error.

8. Show that

$$\int_0^\infty e^{-x^2} dx = x \int_0^\infty e^{-x^2u^2} du;$$

and thence derive the value of the integral.

9. Deduce the formula of reduction ($m$ positive)

$$\int_0^\infty x^m e^{-h^2x^2} dx = \frac{m-1}{2h^2} \int_0^\infty x^{m-2} e^{-h^2x^2} dx;$$

and thence show that ($n$ being a positive integer) the mean value of the $2n$th power of the error is

$$\frac{(2n)!}{2^2\, n!\, h^n},$$

and that the mean absolute value of the $(2n+1)$th power of the error is

$$\frac{n!}{h^{2n+1}\sqrt{\pi}}.$$

10. Show that

$$\operatorname{Erf} t = \int_0^t e^{-t^2} dt = t - \frac{t^3}{3} + \frac{1}{2!}\frac{t^5}{5} - \frac{1}{3!}\frac{t^7}{7} + \cdots$$

11. Deduce the formula of reduction ($n$ positive)

$$\int_t^\infty t^{-n} e^{-t^2} dt = \frac{e^{-t^2}}{2t^{n+1}} - \frac{n+1}{2}\int_t^\infty t^{-(n+2)} e^{-t^2} dt;$$

and thence show that

$$\operatorname{Erfc} t = \int_t^\infty e^{-t^2} dt = \frac{e^{-t^2}}{2t}\left(1 - \frac{1}{2t^2} + \frac{1\cdot 3}{2^2 t^4} - \frac{1\cdot 3\cdot 5}{2^3 t^6} + \cdots\right).$$

12. Find the probability that the deviation of a shot shall exceed $2a$. $\frac{1}{16}$.

13. Find the probability that a shot shall fall within the circle whose radius is $\varepsilon$. $1 - e^{-\frac{1}{8}} = 0.3935$.

14. A marksman shoots 500 times at a target; if his skill is such that when errors are measured in feet, $h = 1$, what is the number of bullet marks between two circles described from the centre with radii 1 and 2 feet? 175.

15. If errors are measured in inches in example 14, what are the values of $h$ and of $a$? $\frac{1}{12}$, 9.99.

16. An archer is observed to plant 9 per cent of his arrows within a circle one foot in diameter; what is the diameter of a target which he might make an even bet to hit at the first shot? 2 ft. 8½ in.

17. A hits a target 3 feet in diameter 51 times out of 79 shots; B hits one 2 feet in diameter 39 times out of 87 shots. Find the diameters of the targets that each can make an even wager to hit at the first shot. For A, 2.45 feet; for B, 2.16 feet.

18. In example 17, what are the odds that B will hit A's probable circle at the first shot? About 59 to 41.

19. If the circular target which a marksman has an even chance of hitting be divided by circumferences cutting the radius into four equal parts, how many shots out of 1000 will fall in the respective areas? 42, 117, 164, 177.

20. A circular target 32 inches in diameter is divided into rings by circumferences cutting the radius into four equal parts. The number of shots out of 1000 which fell in the several areas were 31, 89, 121, 141; what are the respective values of $a$ in inches determined from the numbers of shots in the several circles? 18.764, 18.628, 19.025, 19.202.

21. Find the probability of hitting a square target circumscribing the circle whose radius is $a$. .5790.

22. If several shots be fired at a wafer on a wall and the wafer be subsequently removed, show that the centre of gravity of the shot marks is the most probable position of the wafer.

# V.

## The Combination of Observations and Probable Accuracy of the Results.

### *The Probability of the Arithmetical Mean.*

**62.** We have seen that, in accordance with the law of facility which we have adopted, the best result of the combination of a number of equally good observations is their arithmetical mean. We have next to determine the probable accuracy of this result, and then to consider the best method of combining observations of unequal precision.

Let there be $n$ observations, the law of facility of error for each of which is

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2(x-a)^2}, \quad \ldots \ldots \quad (1)$$

$a$ being the true value of the observed quantity, and $x_1, x_2 \ldots x_n$ the observed values. Then the value of $P$, equation (2), Art. 36, becomes

$$P = \frac{h^n}{\pi^{\frac{1}{2}n}} e^{-h^2\Sigma(x-a)^2} \Delta x^n; \quad \ldots \ldots \quad (2)$$

and, as shown in Art. 30, the probabilities of the different hypotheses which we can make as to the value of $a$ are proportional to the corresponding values of $P$.

**63.** Let us now take $a$ to denote the arithmetical mean, and put $a - \delta$ for the true value, so that $\delta$ is the error of the arithmetical mean; then denoting the residual by $v$, the true error will be $x - a + \delta = v + \delta$. It was shown in Art. 43 that

$$\Sigma(v+\delta)^2 = \Sigma v^2 + n\delta^2;$$

hence the general value of $P$ must now be written

$$P = \frac{h^n}{\pi^{\frac{1}{2}n}} e^{-h^2[\Sigma v^2 + n\delta^2]} \Delta x^n, \quad \ldots \ldots \quad (3)$$

and the value expressed by equation (2) is now the maximum value, corresponding to $\delta = 0$. Distinguishing this value by the symbol $P_0$, equation (3) may be written

$$P = P_0 e^{-nh^2\delta^2} \quad . \quad . \quad . \quad (4)$$

Since the probability of $\delta$, with respect to the error of our final determination, is proportional to $P$, and $P_0$ is independent of $\delta$, equation (4) shows that the arithmetical mean has a law of probability which is identical with that which we have adopted in equation (1) for the single observations, except that $nh^2$ takes the place of $h^2$. Thus, denoting by $y_0$ the facility of error in the arithmetical mean, we have

$$y_0 = \frac{h\sqrt{n}}{\sqrt{\pi}} e^{-nh^2\delta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The fact that the assumption of the law (1) for a single observation implies a law of the same form for the final value determined from the combined observations is one of the confirmations of this law alluded to in Art. 40.*

**64.** Equation (5) of the preceding article shows that the arithmetical mean of $n$ observations may be regarded as an observation made with a more precise instrument, the new measure of precision being found by multiplying that of the single observations by $\sqrt{n}$. Since $hr$ is constant when $r$ represents any one of the measures of risk, we have for *the probable error of the arithmetical mean*,

$$r_0 = \frac{r}{\sqrt{n}},$$

*In general, an assumed law, $y = \phi(x)$, of facility of error for the single observations would produce a law of a different form for the result determined from $n$ observations. Laplace has shown that whatever be the form of $\phi$ for the single observations, the law of facility of error in the arithmetical mean approaches indefinitely to

$$y = ce^{-h^2x^2}$$

as a limiting form, when $n$ is increased without limit. See the memoir "On the Law of Facility of Errors of Observation, and on the Method of Least Squares," by J. W. L. Glaisher, *Memoirs Royal Ast. Soc.*, vol. xxxix pp. 104, 105.

and the same relation holds in the case of either of the other measures of risk.

Thus, for example, it is necessary to take four observations in order to double the precision, or reduce the risk of error to one half its original value.

The probable error of a final result is frequently written after it with the sign $\pm$. Thus, if the final determination of an angle is given as $36° 42'.3 \pm 1'.22$, the meaning is that the true value of the angle is exactly as likely to lie between the limits thus assigned (that is, between $36° 41'.08$ and $36° 43'.52$) as it is to lie outside of these limits.

*The Combination of Observations of Unequal Precision.*

**65.** When the observations are not equally good, let $h_1, h_2, \ldots h_n$ be their respective measures of precision; so that, $a$ being the true value, the facility of error of $x_1$ is

$$y_1 = \frac{h_1}{\sqrt{\pi}} e^{-h_1^2(x_1-a)^2},$$

that of $x_2$ is

$$y_2 = \frac{h_2}{\sqrt{\pi}} e^{-h_2^2(x_2-a)^2},$$

and so on. The value of $P$, Art. 36, which expresses the probability of the given system of observed values on the hypothesis of a given value of $a$, now becomes

$$P = \frac{h_1 h_2 \ldots h_n}{\pi^{\frac{1}{2}n}} e^{-\Sigma h^2(x-a)^2} \Delta x_1 \Delta x_2 \ldots \Delta x_n; \quad . \quad (1)$$

and, as before, the probabilities of different values of $a$ are proportional to the values they give to $P$.

It follows that that value of $a$ is most probable which makes $\Sigma h^2(x-a)^2$ or

$$h_1^2(x_1-a)^2 + h_2^2(x_2-a)^2 + \ldots + h_n^2(x_n-a)^2 = \text{a minimum.} \quad (2)$$

In other words, if the error of each observation be multiplied by the corresponding measure of precision, so as to reduce the errors

to the same relative value (see Art. 47), *it is necessary that the sum of the squares of the reduced errors should be a minimum.* This is, in fact, the more general statement of the principle of Least Squares.

Differentiating with respect to $a$, we have

$$h_1^2(x_1 - a) + h_2^2(x_2 - a) + \ldots + h_n^2(x_n - a) = 0; \quad (3)$$

and the value of $a$ determined from this equation is

$$a = \frac{h_1^2 x_1 + h_2^2 x_2 + \ldots + h_n^2 x_n}{h_1^2 + h_2^2 + \ldots + h_n^2} = \frac{\Sigma h^2 x}{\Sigma h^2}, \quad \ldots \quad (4)$$

which is therefore the most probable value of $a$ which can be derived from the $n$ observations.

### *Weights and Measures of Precision.*

**66.** The value of $a$ found above is in fact the weighted arithmetical mean of the observed values (see Art. 11), when the respective values of $h^2$ are taken as the weights. But, since the weights are numbers with whose ratios only we are concerned, we may use any proportional numbers $p_1, p_2, \ldots p_n$, in place of the values of $h$. Thus putting

$$h_1^2 = p_1 h^2, \quad h_2^2 = p_2 h^2, \quad \ldots \quad h_n^2 = p_n h^2, \quad \ldots \quad (5)$$

equation (4) may be written

$$a = \frac{p_1 x_1 + p_2 x_2 + \ldots + p_n x_n}{p_1 + p_2 + \ldots + p_n} = \frac{\Sigma p x}{\Sigma p}. \quad \ldots \quad (6)$$

Hence the most probable value which can be derived from the $n$ observations is *the weighted arithmetical mean, the weights of the observations being proportional to the squares of their measures of precision.*

The quantity $h$ in equations (5) is the measure of precision of an observation whose weight is unity. It is immaterial whether such an observation actually exists among the $n$ observations or not.

If each of the observations has the weight unity, $\Sigma p$ takes the value $n$, and the value of $a$ becomes the ordinary arithmetical mean.

*The Probability of the Weighted Mean.*

**67.** Let us now, employing $a$ to denote the value determined above, put $a + \delta$ in place of $a$ in the value of $P$, so that $\delta$ represents the error in our final determination of $a$. Then, writing $v$ for the residual, we have, as in Art. 63, to replace $x - a$ by $v + \delta$. The value of $P$, equation (1), Art. 65, thus becomes

$$P = \frac{h_1 h_2 \ldots h_n}{\pi^{\frac{1}{2}n}} e^{-\Sigma h^2 (v+\delta)^2} \Delta x_1 \Delta x_2 \ldots \Delta x_n$$

Now, by equation (3), $\Sigma h^2 v = 0$, therefore

$$\Sigma h^2 (v + \delta)^2 = \Sigma h^2 v^2 + \delta^2 \Sigma h^2 \cdot$$

substituting, we obtain

$$P = \frac{h_1 h_2 \ldots h_n}{\pi^{\frac{1}{2}n}} e^{-\Sigma h^2 v^2} e^{-\delta^2 \Sigma h^2} \Delta x_1 \Delta x_2 \ldots \Delta x_n.$$

Hence, putting $P_0$ for the value assumed by $P$ when $\delta = 0$, we have

$$P = P_0 e^{-\delta^2 (h_1^2 + h_2^2 + \ldots + h_n^2)}.$$

Since the probability of $\delta$ is proportional to $P$, it follows, as in Art. 63, that the law of facility of the mean is of the same form as those of the separate observations, the square of the new measure of precision being the sum of the squares of those of the separate observations. Denoting the facility of error in the weighted mean by $y_0$, and employing the notation of Art. 66, we have therefore

$$y_0 = \frac{h \sqrt{\Sigma p}}{\sqrt{\pi}} e^{-\delta^2 h^2 \Sigma p},$$

in which $h$ is the measure of precision of an observation whose

weight is unity. When the weights are all equal, this formula becomes identical with that of Art. 63.

**68.** The weight of the mean is defined in Art. 12 to be $\Sigma p$, the sum of the weights of the constituent observations. Hence the value of $y_0$ found above shows that, *in comparing the final result with any single observation*, as well as in comparing the observations with one another, *the measures of precision are proportional to the square roots of the weights.*

The probable error being inversely proportional to $h$, it follows that, $r$ representing the probable error of an observation whose weight is unity, and $r_0$ that of the mean whose weight is $\Sigma p$, we shall have

$$r_0 = \frac{r}{\sqrt{\Sigma p}}.$$

This result includes that of Art. 64, and, like it, is applicable to either of the measures of risk.

*The Most Probable Value of h derivable from a System of Observations.*

**69.** Substituting the values of $h_1, h_2, \ldots h_n$ in terms of the weights, equations (5), Art. 66, the value of $P$, equation (1), Art. 65, becomes

$$P = \frac{\sqrt{(p_1 p_2 \ldots p_n)}}{\pi^{\frac{1}{2}n}} h^n e^{-h^2 \Sigma p (x-a)^2} \Delta x_1 \Delta x_2 \ldots \Delta x_n. \quad (1)$$

The same principle which we have employed to determine the most probable value of the observed quantity serves to determine the most probable value of $h$. Thus the most probable value of $h$ is that which gives the greatest value to $P$, or, omitting factors independent of $h$, to the expression

$$h^n e^{-h^2 \Sigma p (x-a)^2}.$$

Putting the derivative of this expression equal to zero, we have

$$e^{-h^2 \Sigma p (x-a)^2} [n h^{n-1} - 2h^{n+1} \Sigma p (x-a)^2] = 0;$$

whence

$$h = \sqrt{\frac{n}{2\Sigma p(x-a)^2}}, \quad \ldots\ldots \quad (2)$$

in which $a$ denotes the true value of the observed quantity.

**70.** Equation (2) may be written

$$\frac{\Sigma p(x-a)^2}{n} = \frac{1}{2h^2} \ldots\ldots\ldots \quad (1)$$

When the observations are all made under the same circumstances, so that we may put

$$p_1 = p_2 = \ldots = p_n = 1,$$

the equation becomes

$$\frac{\Sigma(x-a)^2}{n} = \frac{1}{2h^2}, \quad \ldots\ldots \quad (2)$$

in which $h$ denotes the measure of precision of each of the observations. The second member of this equation is the value of $\varepsilon^2$, the square of the "mean error," which was defined in Art. 50 as the mean value of the square of the error, having regard to its probability in a system of observations whose measure of precision is $h$. In other words, it is the mean squared error in an unlimited number of observations made under the given circumstances of observation.

On the other hand, the first member of equation (2) is the actual mean squared error for the $n$ given observations. The square root of this quantity may be called the *observational* value of the mean error, in distinction from the *theoretical* value, $\varepsilon$, which is a fixed function of $h$.

Thus the equation asserts that the most probable value of $h$ is found *by assuming the theoretical value of the mean error to be the same as its observational value.* In other words, it is a consequence of the accepted law of facility that the measure of precision of a set of observations equally good is proportional to the reciprocal of the mean error as determined from the observations themselves.

*Formulæ for the Mean and Probable Errors.*

**71.** The quantity $\Sigma p(x-a)^2$ in the value of $h$, equation (2), Art. 69, is the sum of the weighted squares of the actual errors of the observed values $x_1, x_2, \ldots x_n$. Now, when $a$ denotes the weighted arithmetical mean, $x-a$ must be replaced by $v+\delta$, as in Art. 67, and

$$\Sigma p(v+\delta)^2 = \Sigma pv^2 + \delta^2 \Sigma p. \quad \ldots \quad (1)$$

The value of $\delta$, which is the error of the arithmetical mean, is of course unknown; it may be either positive or negative, but, since $\delta^2$ is essentially positive, the true value of $\Sigma p(x-a)^2$ always exceeds $\Sigma pv^2$. The best correction we can apply to the approximate value $\Sigma pv^2$ is found by giving to $\delta^2$ in equation (1) its mean value; for, by adopting this as a general rule we shall commit the least error in the long run. Now we have seen in Art. 67 that $\delta$ follows a law of probability of the usual form in which the measure of precision is $h\sqrt{\Sigma p}$, hence the mean value of $\delta^2$ is the same as the mean squared error found in Art. 50, except that $h$ is changed to $h\sqrt{\Sigma p}$. That is to say, the mean value of $\delta^2$ is

$$\frac{1}{2h^2\Sigma p}.$$

Putting this in place of $\delta^2$ in equation (1) we have

$$\Sigma p(v+\delta)^2 = \Sigma pv^2 + \frac{1}{2h^2}. \quad \ldots \quad (2)$$

Equation (2), Art. 69, may be written in the form

$$\frac{n}{h^2} = 2\Sigma p(x-a)^2,$$

and, employing the value just determined, we have

$$\frac{n}{h^2} = 2\Sigma pv^2 + \frac{1}{h^2};$$

whence we derive

$$h = \sqrt{\frac{n-1}{2\Sigma pv^2}} \quad \ldots \quad (3)$$

for the most probable value of $h$ for an observation of weight unity.

**72.** The resulting value of the mean error of an observation whose weight is unity is

$$\epsilon = \frac{1}{h\sqrt{2}} = \sqrt{\frac{\Sigma pv^2}{n-1}}, \quad \ldots\ldots \quad (1)$$

and by Art. 68, the mean error of the arithmetical mean whose weight is $\Sigma p$ is

$$\epsilon_0 = \sqrt{\frac{\Sigma pv^2}{(n-1)\Sigma p}}. \quad \ldots\ldots \quad (2)$$

Again, the value of the probable error of an observation whose weight is unity is

$$r = \frac{\rho}{h} = \rho\sqrt{2}\sqrt{\frac{\Sigma pv^2}{n-1}} = 0.6745\sqrt{\frac{\Sigma pv^2}{n-1}}, \quad \ldots \quad (3)$$

and that of the weighted arithmetical mean is

$$r_0 = 0.6745\sqrt{\frac{\Sigma pv^2}{(n-1)\Sigma p}}. \quad \ldots\ldots \quad (4)$$

The constant 0.6745 is the reciprocal of that which occurs in equation (2), Art. 51.

For a set of equally good observations we have, by putting $p_1 = p_2 = \ldots = p_n = 1$,

$$r = 0.6745\sqrt{\frac{\Sigma v^2}{n-1}} \quad \ldots\ldots\ldots \quad (5)$$

for the probable error of a single observation, and

$$r_0 = 0.6745\sqrt{\frac{\Sigma v^2}{n(n-1)}} \quad \ldots\ldots\ldots \quad (6)$$

for the probable error of the simple arithmetical mean.

*The Most Probable Value of h in Target Practice.*

73. We have seen in Art. 55 that in target practice the probability of hitting an elementary area $a$, situated at the distance $r$ from the point aimed at, is

$$\frac{h^2}{\pi} e^{-h^2 r^2} a.$$

Suppose that $n$ shots have been made, the first falling upon the area $a_1$, the second upon $a_2$, and so on; then, before the shots were made, the probability that the shots should fall upon these areas in the given succession is

$$P = \frac{h^{2n}}{\pi^n} e^{-h^2 \Sigma r^2} a_1 a_2 \ldots a_n.$$

Hence, the shots having been made, the probabilities of different values of $h$ are proportional to the values they give to the expression

$$h^{2n} e^{-h^2 \Sigma r^2}.$$

Making this function of $h$ a maximum, we have

$$e^{-h^2 \Sigma r^2} [2nh^{2n-1} - 2h^{n+1} \Sigma r^2] = 0,$$

whence we have, for the most probable value of $h$,

$$h = \sqrt{\frac{n}{\Sigma r^2}},$$

the value quoted in Art. 61.

74. The value of $\varepsilon^2$ hence derived is

$$\varepsilon^2 = \frac{\Sigma r^2}{2n} = \frac{\Sigma x^2 + \Sigma y^2}{2n},$$

where $\varepsilon$ is the mean error for the component deviations, which are the values of $x$ and $y$ respectively. The values of $\varepsilon^2$ as determined from the lateral and vertical deviations respectively, are

$$\epsilon^2 = \frac{\Sigma x^2}{n}, \quad \epsilon^2 = \frac{\Sigma y^2}{n}.$$

Thus the value of $\epsilon^2$, which we have derived from the total deviations, or values of $r$, is the mean of its most probable values as separately derived from the two classes of component deviations.

It will be noticed that neither of the quantities $\Sigma x^2$, $\Sigma y^2$ or $\Sigma r^2$ needs to be corrected as in Art. 71, because we are here dealing with actual errors and not with residuals.*

*The Computation of the Probable Error.*

**75.** The annexed table gives an example of the application of formulæ (5) and (6), Art. 72. The seventeen values of $x$ in

| $x$ | $v$ | $v^2$ |
|---|---|---|
| 4·524 | + .0185 | .00034225 |
| 4·500 | − .0055 | 3025 |
| 4·515 | + .0095 | 9025 |
| 4·508 | + .0025 | 625 |
| 4·513 | + .0075 | 5625 |
| 4·511 | + .0055 | 3025 |
| 4·497 | − .0085 | 7225 |
| 4·507 | + .0015 | 225 |
| 4·501 | − .0045 | 2025 |
| 4·502 | − .0035 | 1225 |
| 4·485 | − .0205 | 42025 |
| 4·519 | + .0135 | 18225 |
| 4·517 | + .0115 | 13225 |
| 4·504 | − .0015 | 225 |
| 4·493 | − .0125 | 15625 |
| 4·492 | − .0135 | 18225 |
| 4·505 | − .0005 | 25 |
| $a = 4.505\frac{8}{17} = 4.5055$ | | $\Sigma v^2 = .00173825$ |

*If the position of the point aimed at had been inferred from the shot marks, as in example 22 of the preceding section, it would have been necessary to change $n$ into $n - 1$, as in the case of errors of observation. So also this change should be made when the errors employed are measured from the mean point of impact, as in testing pieces of ordnance.

the first column are independent measurements of the same quantity made by Prof. Rowland for the purpose of determining a certain wave length. At the foot of the column is the arithmetical mean of the seventeen observations. The second column contains the residuals found by subtracting this from the separate observations. The values of $v^2$ in the third column are taken from a table of squares, and their sum is written at the foot of the column. Dividing this by 16, the value of $n - 1$, we find

$$\frac{\Sigma v^2}{n-1} = 0.00010864,$$

and taking the square root,

$$\epsilon = 0.01042.$$

Multiplying by the constant 0.6745 we have

$$r = 0.00703$$

for the probable error of a single observation.

Again, dividing by $\sqrt{17}$, we have

$$r_0 = 0.00171$$

for the probable error of the final determination, which may therefore be written

$$x = 4.5055 \pm 0.0017.$$

It will be noticed that nine of the residuals are numerically less and eight are numerically greater than the value we have found for the probable error of a single observation.

**76.** The equation

$$\Sigma(v + \delta)^2 = \Sigma v^2 + n\delta^2,$$

derived in Art. 43, enables us to abridge somewhat the computation of $\Sigma v^2$, and to reduce the extent to which a table of squares is needed. Thus, if we use the value of $a$ to three places of decimals, namely $a = 4.505$, in forming the values of

$v$, each of these quantities will be algebraically greater than it should be by $\frac{8}{17}$ of a unit in the third decimal place. Putting

$$\delta = \tfrac{8}{17}, \quad n\delta^2 = \tfrac{64}{17} = 3\tfrac{13}{17};$$

hence $\Sigma v^2$, as found on this supposition, will be too great by $3\frac{13}{17}$ of a unit in the sixth decimal place. The columns headed $v$ and $v^2$ would then stand as follows:

| $v$ | $v^2$ |
|---|---|
| + .019 | .000361 |
| − .005 | 25 |
| + .010 | 100 |
| + .003 | 9 |
| + .008 | 64 |
| + .006 | 36 |
| − .008 | 64 |
| + .002 | 4 |
| − .004 | 16 |
| − .003 | 9 |
| − .020 | 400 |
| + .014 | 196 |
| + .012 | 144 |
| − .001 | 1 |
| − .012 | 144 |
| − .013 | 169 |
| .000 | 0 |

$$\Sigma(v + \delta)^2 = .001742$$

and making the correction found above, we have

$$\Sigma v^2 = .001738\tfrac{4}{17},$$

which is the exact value.

The smallness of the correction is due to the fact that $\Sigma v^2$ is a minimum value. The correction might have been neglected, being, in this case, only about $\frac{1}{30}$ of the correction made in the formula on account of the mean value of the unknown error in the arithmetical mean.

**77.** As an example of the application of the formulæ involving weights, let us suppose that instead of the seventeen observations in the preceding article we were given only the means of certain groups into which the seventeen observations may be separated. These means we have seen may be regarded as observations having weights equal to the respective numbers of observations from which they are derived. The annexed table presents the

| $p$ | $x$ | $v$ | $v^2$ | |
|---|---|---|---|---|
| 2 | 4.512 | + .0065 | .00004225 | .00008450 |
| 1 | 4.515 | + .0095 | 9025 | 9025 |
| 4 | 4.507 | + .0015 | 225 | 900 |
| 3 | 4.503 | − .0025 | 625 | 1875 |
| 2 | 4.502 | − .0035 | 1225 | 2450 |
| 2 | 4.511 | + .0055 | 3025 | 6050 |
| 3 | 4.497 | − .0085 | 7225 | 21675 |
| | $a = 4.5055$ | | | $\Sigma pv^2 = .00050425$ |

data in such a form, the first value of $x$ being the mean of the first two values in the preceding table, the next being the third observation, the next the mean of the following four, and so on. The weighted mean of the present seven values of $x$ of course agrees with the final value before found. The values of $v$ and of $v^2$ are formed as before, and the values of $pv^2$ are given in the last column, at the foot of which is the value of $\Sigma pv^2$. Dividing this by 6, the present value of $n - 1$, we find

$$\frac{\Sigma pv^2}{n-1} = 0.00008304,$$

and, multiplying the square root of this by 0.6745, the value of the probable error of an observation whose weight is unity is

$$r = 0.00615.$$

The probable error of the weighted mean found by dividing this by $\sqrt{17}$, the value of $\sqrt{\Sigma p}$, is

$$r_0 = 0.00149.$$

78. The value of $r$ found above corresponds to a single observation of the set given in Art. 75. It differs considerably from the value found in that article. The discrepancy is due to the fact that in Art. 76 we did not use all the data given in Art. 75, and it is not to be expected that the most probable value of $h$ which can be deduced from the imperfect data should agree with that deduced from the more complete data. In one case we have seventeen discrepancies from the arithmetical mean, due to accidental errors, upon which to base an estimate of the precision of the observations; in the other case we have but seven discrepancies. The result in the former case is of course more trustworthy; and in general, the larger the value of $n$, the more confidence can we place in our estimate of the measures of precision.

79. It should be noticed particularly that the weighted observations in Art. 76 are not equivalent to a set of seventeen observations of which two are equal to the first value of $x$, one to the second, four to the third, and so on, *except in the sense of giving the same mean value.* Compare Art. 10. Such a set would exhibit discrepancies very much smaller on the whole than those of the seventeen observations in Art. 75. Accordingly, the value of $\varepsilon^2$ in the supposed case would be very much smaller than that found above for the weighted observations. The value of $\Sigma v^2$ would in fact be the same as that of $\Sigma pv^2$ in Art. 76, but it would be divided by 16 instead of by 6.

The approximate equality of the results in Art. 75 and Art. 76 is due to the fact that the $v^2$'s, of which seventeen exist in each sum, are on the average very much diminished* when the mean of a group is substituted for the separate observations, and this

* The amount of this diminution is, however, largely a matter of chance. For example, if we had taken the seven groups in such a manner that the successive values of $p$ were 2, 3, 2, 4, 2, 1, 3, we should have found

$$r = 0.00833,$$

differing in excess from that of Art. 75 still more than that obtained above does in defect.

makes up for the change in the denominator by the decrease in the value of $n$.

**80.** Different weights are frequently assigned to observations made under different circumstances, according to the judgment of the observer. Thus an astronomer may regard an observation made when the atmosphere is exceptionally clear as worth two of those made under ordinary circumstances. Regarding the latter as standard observations having the weight unity, he will then assign the weight 2 to the former. As explained in the preceding article this is not equivalent to recording two standard observations, each giving the observed value. The latter procedure would lead to an erroneous estimate of the degree of accuracy attained.

### *The Values of h and r derived from the Mean Absolute Error.*

**81.** The mean absolute error $\eta$ is a fixed function of $h$, viz:

$$\eta = \frac{1}{h\sqrt{\pi}}; \quad \ldots\ldots\ldots \quad (1)$$

hence, if we were able to determine it independently, we should have a means of finding the value of $h$, and consequently that of $r$.

In the case of $n$ equally good observations, let $[x - a]$ denote the numerical value of an error taken as positive, then

$$\frac{\Sigma[x - a]}{n} \quad \ldots\ldots\ldots \quad (2)$$

is the arithmetical mean of the absolute values of the $n$ actual errors. This may be called the *observational value* of the mean absolute error in distinction from the *theoretic value* given in equation (1), which is the value of this mean in accordance with the law of probability, when the measure of precision is $h$.

If we assume these values to be equal, we obtain

$$\frac{\Sigma[x - a]}{n} = \frac{1}{h\sqrt{\pi}},$$

whence

$$h = \frac{n}{\Sigma[x-a]\sqrt{\pi}}, \quad \ldots \ldots \quad (3)$$

and

$$r = \frac{\rho}{h} = \rho\sqrt{\pi}\frac{\Sigma[x-a]}{n} \quad \ldots \ldots \quad (4)$$

If in this formula we put for $a$ the arithmetical mean, so that $\Sigma[x-a]$ becomes $\Sigma[v]$, it gives the *apparent probable error*, that is, the value $r$ would have if the arithmetical mean were known to be the true value of $x$. Denoting this by $r'$, we have then

$$r' = \rho\sqrt{\pi}\frac{\Sigma[v]}{n} = 0.8453\frac{\Sigma[v]}{n}. \quad \ldots \quad (5)$$

**82.** It is obvious from Arts. 71 and 72 that the values of $r'$ and $r$ as derived from the square of the residuals are

$$r' = 0.6745\sqrt{\frac{\Sigma v^2}{n}}, \qquad r = 0.6745\sqrt{\frac{\Sigma v^2}{n-1}},$$

so that

$$r : r' = \sqrt{n} : \sqrt{(n-1)}.^* \quad \ldots \ldots \quad (6)$$

* This relation between the apparent and the real probable error is derived directly by C. A. F. Peters (*Berliner Astronomisches Nachrichten*, 1856, vol. xliv. p. 29) as follows: If $e_1, e_2, \ldots e_n$ are the true errors, that of the arithmetical mean is

$$\delta = \frac{1}{n}(e_1 + e_2 + \ldots + e_n),$$

then

$$v_1 = e_1 - \delta = \frac{n-1}{n}e_1 - \frac{1}{n}e_2 - \ldots - \frac{1}{n}e_n, \quad \text{etc.}$$

Since $r$ is the probable error of each $e$, and $r'$ that of each $v$, the formula for the probable error of a linear function of independent quantities (see Art. 89) gives

$$r'^2 = \left[\left(\frac{n-1}{n}\right)^2 + (n-1)\frac{1}{n^2}\right]r^2 = \frac{n-1}{n}r^2.$$

This result is used by Peters to establish the formula derived above, but it may also be used in place of the method of Art. 71 for the correction of the apparent value of $r$ in terms of $\Sigma v^2$.

Combining this result with equation (5) we have

$$r = 0.8453\frac{\Sigma[v]}{\sqrt{[n(n-1)]}}, \quad \ldots \ldots \quad (7)$$

and hence, for the probable error of the arithmetical mean,

$$r_0 = 0.8453\frac{\Sigma[v]}{n\sqrt{(n-1)}}. \quad \ldots \ldots \quad (8)$$

As an illustration, let us apply these formulæ to the observations given in Art. 75, for which we find $\Sigma[v] = 0.1405$. Substituting this value, and putting $n = 17$, we find

$$r = 0.00720, \qquad r_0 = 0.00175.$$

These values agree closely with those derived in Art. 75 from the formulæ involving $\Sigma v^2$, which indeed give the most probable values of $r$ and $r_0$, but involve much more numerical work, especially when $n$ is large.

**83.** In order to adapt the formulæ of Art. 82 to the case of weighted observations, it is necessary to reduce the errors to the same scale; in other words, to make them proportional to the reduced errors or values of $t$, see Art. 47. Since the measures of precision are proportional to the square roots of the weights, this is effected by multiplying each error by the square root of the corresponding weight. The products may be regarded as errors belonging to the same system, namely, that which corresponds to the weight unity.

Hence equation (7) gives for the probable error of an observation whose weight is unity

$$r = 0.8453\frac{\Sigma[v\sqrt{p}]}{\sqrt{[n(n-1)]}},$$

and for the probable error of the weighted arithmetical mean we have

$$r_0 = 0.8453\frac{\Sigma[v\sqrt{p}]}{\sqrt{[n(n-1)\Sigma p]}}.$$

*Examples.*

1. A line is measured five times and the probable error of the mean is .016 of a foot. How many additional measurements of the same precision are required in order to reduce the probable error of the determination to .004 of a foot? 75.

2. It is required to determine an angle with a probable error less than 0″.25. The mean of twenty measurements gives a probable error of 0″.38; how many additional measurements are necessary? 27.

3. If the probable error of each of two like measurements of a foot bar is .00477 of an inch, what is the probable error of their mean? .00337.

4. Ten measurements of the density of a body made with equal precision gave the following results:

| | | | | |
|---|---|---|---|---|
| 9.662, | 9.664, | 9.677, | 9.663, | 9.645, |
| 9.673, | 9.659, | 9.662, | 9.680, | 9.654. |

What is the probable value of the density of the body and the probable error of that value? 9.6639 ± .0022.

5. Forty micrometric measurements of the error of position of a division line upon a standard scale gave the following results:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3.68 | 5.08 | 2.81 | 4.43 | 5.48 | 4.21 | 3.28 | 5.21 |
| 3.11 | 2.95 | 4.65 | 3.43 | 3.76 | 5.23 | 3.78 | 4.43 |
| 4.76 | 6.35 | 3.27 | 3.26 | 4.59 | 4.45 | 3.22 | 2.28 |
| 2.75 | 3.78 | 4.08 | 2.48 | 2.64 | 3.95 | 3.98 | 4.10 |
| 4.15 | 4.49 | 4.51 | 4.84 | 2.98 | 2.66 | 3.91 | 4.18 |

Find the probable value of the quantity measured and its probable error. 3.930 ± 0.097.

6. In the preceding example what is the probable error of a single observed quantity: 1°, by the formula involving the squares of the errors; 2°, by that involving the absolute errors? 1°, $r = 0.616$; 2°, $r = 0.618$.

7. An angle in the primary triangulation of the U. S. Coast Survey was measured twenty-four times with the following results:

| | | | | | |
|---|---|---|---|---|---|
| 116° 43′ 44″.45 | 49.20 | 51.05 | 51.75 | 51.05 | 49.25 |
| 50 .55 | 48.85 | 47.85 | 49.00 | 51.70 | 46.75 |
| 50 .95 | 47.40 | 50.60 | 52.35 | 49.05 | 49.25 |
| 48 .90 | 47.75 | 48.45 | 51.30 | 50.55 | 53.40 |

Find the probable error of a single measurement, and the final determination of the angle. 1″.35: 116° 43′ 49″.64 ± 0″.28.

8. In example 7, taking the means of the six groups of four observations each, determine the probable error of the first of these means: 1°, considered as a measurement of four times the weight of those in example 7; 2°, directly as one of six observations of equal weight; 3°, as a determination from its four constituents. 1°, 0″.67; 2°, 0″.72; 3°, 1″.00.

9. An interval of 600 units as determined by a micrometer was forty times measured to determine the error in the pitch of the screw, with the following results:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 600.0 | 604.8 | 600.7 | 601.4 | 602.0 | 602.6 | 600.0 | 602.4 |
| 599.7 | 606.1 | 602.4 | 603.4 | 602.7 | 602.7 | 600.7 | 602.4 |
| 599.5 | 604.7 | 601.6 | 603.1 | 603.7 | 600.9 | 601.4 | 602.1 |
| 604.6 | 602.1 | 601.7 | 601.8 | 602.1 | 601.4 | 602.9 | 603.6 |
| 603.9 | 602.2 | 601.4 | 600.6 | 602.3 | 600.8 | 602.9 | 603.6 |

Find the probable value of the interval and its probable error. 602.22 ± 0.157.

# VI.

## The Facility of Error in a Function of One or More Observed Quantities.

### *The Linear Function of a Single Observed Quantity.*

**84.** If the value of an observed quantity $X$ be subject to an error $x$, the value of a given function of $X$, say $Z=f(X)$, will be subject to a corresponding error $z$. Assuming $x$ to follow the usual law of facility, $h$ being the measure of precision and $r$ the probable error, we have now to determine the law of facility of $z$, for any form of the function $f$.

Let us first consider the linear function

$$Z = mX + b,$$

where $m$ and $b$ are constants. The case is obviously the same as that of the simple multiple $mX$, the relation between the corresponding errors being

$$z = mx.$$

The probability that the error $z$ falls between $z$ and $z + dz$ is the same as the probability that $x$ falls between $x$ and $x + dx$, namely,

$$\frac{h}{\sqrt{\pi}} e^{-h^2x^2} dx.$$

Expressing this in terms of $z$, it becomes

$$\frac{h}{\sqrt{\pi}} e^{-\frac{h^2z^2}{m^2}} \frac{dz}{m},$$

or, putting $\frac{h}{m} = H$,

$$\frac{H}{\sqrt{\pi}} e^{-H^2z^2} dz.$$

Thus the law of facility for $Z$ is of the same form as that for $X$,

the measure of precision being found by dividing that of $X$ by $m$; and, denoting the probable error of $Z$ by $R$, we have (since probable errors are inversely as the measures of precision)

$$R = mr,$$

and the same relation holds between either of the other measures of the risk of error.

The curves of facility for $X$ and $Z$ are related in the same manner as those drawn in Fig. 4, page 30, and the process of passing from one to the other is that described in Art. 46; that is to say, the abscissas which represent the errors are multiplied by $m$, and then the ordinates are divided by $m$, so that the areas standing upon the corresponding bases $dx$ and $dz$ shall remain equal.

*Non-Linear Functions of a Single Observed Quantity.*

**85.** A non-linear function of an observed quantity subject to the usual law of facility does not strictly follow a law of facility of the same form. If, however, as is usually the case, the error $x$ is very small, any function of the observed quantity will very nearly follow a law of the usual form. Let $a$ be the true value of the observed quantity, then

$$X = a + x,$$

and

$$Z = f(X) = f(a + x).$$

Expanding by Taylor's Theorem, and neglecting the higher powers of $x$,* we may take

$$Z = f(a) + xf'(a),$$

which is of the linear form. Hence we may regard $Z$ as subject to the usual law of facility, its probable error being

$$R = rf'(a),$$

or, putting the observed value in place of $a$,

$$R = rf'(X).$$

* The ratio of the square of the error to the error itself is the value of the error considered as a number, and it is this numerical value which must be small.

*The Facility of Error in the Sum or Difference of Two Observed Quantities.*

**86.** Let $X$ and $Y$ be two observed quantities subject to the usual law of facility of error, their measures of precision being $h$ and $k$ respectively. If

$$Z = X + Y,$$

the relation between the errors of $Z$, $X$ and $Y$ is obviously

$$z = x + y.$$

In order to find the facility of $z$, that is, the probability that $z$ shall fall between $z$ and $z + dz$, let us first suppose that $x$ has a definite fixed value. With this hypothesis, the probability in question is the same as the probability that $y$ shall fall between $y$ and $y + dy$, where

$$y = z - x, \quad \text{and} \quad dy = dz.$$

This probability is

$$\frac{k}{\sqrt{\pi}} e^{-k^2y^2} dy, \quad \text{or} \quad \frac{k}{\sqrt{\pi}} e^{-k^2(z-x)^2} dz.$$

Multiplying by the elementary probability of the hypothesis made, which is

$$\frac{h}{\sqrt{\pi}} e^{-h^2x^2} dx,$$

we have

$$\frac{hk}{\pi} e^{-h^2x^2 - k^2(z-x)^2} dz\, dx \quad \ldots\ldots \quad (1)$$

for the probability that the required event (namely, the occurrence of the particular value of $z$) shall happen in this particular way, that is, in connexion with the particular value of $x$. To find the total probability of the event we therefore sum the above expression for all possible values of $x$, thus obtaining

$$\frac{hk}{\pi} \int_{-\infty}^{\infty} e^{-(h^2+k^2)x^2 + 2k^2zx - k^2z^2} dz\, dx. \quad \ldots \quad (2)$$

The exponent of $e$ in this expression may be written

$$-(h^2+k^2)\left(x-\frac{k^2z}{h^2+k^2}\right)^2+\frac{k^4z^2}{h^2+k^2}-k^2z^2;$$

whence, putting $\alpha=\frac{k^2z}{h^2+k^2}$ and

$$H^2=k^2-\frac{k^4}{h^2+k^2}=\frac{h^2k^2}{h^2+k^2}, \quad \ldots \quad (3)$$

the expression (2) becomes

$$\frac{hkdz}{\pi}\,e^{-H^2z^2}\int_{-\infty}^{\infty}e^{-(h^2+k^2)(x-\alpha)^2}dx.$$

Since $\alpha$ is independent of $x$, the value of the integral contained in this expression is, by Art. 39, $\frac{\sqrt{\pi}}{\sqrt{(h^2+k^2)}}$; hence the probability that $z$ shall fall between $z$ and $z+dz$ is

$$\frac{hk}{\sqrt{\pi}\sqrt{(h^2+k^2)}}\,e^{-H^2z^2}dz, \quad \text{or} \quad \frac{H}{\sqrt{\pi}}\,e^{-H^2z^2}dz.$$

**87.** The result just obtained shows that the sum of two quantities subject to the usual law of facility of error is subject to a law of the same form, its measure of precision being determined by equation (3).

Writing equation (3) in the form

$$\frac{1}{H^2}=\frac{1}{h^2}+\frac{1}{k^2},$$

it is evident that, if $r_1$, $r_2$ and $R$ be the probable errors of $X$, $Y$ and $X+Y$, we shall have

$$R^2=r_1^2+r_2^2,$$

the same relation holding in the case of either of the other measures of risk of error.

For the difference

$$Z=X-Y,$$

we have the same result; for the errors of $-Y$ have obviously the same law of facility as those of $Y$.

88. As an illustration, suppose the latitude $\varphi$ and the polar distance $p$ of a circumpolar star to be determined from the altitudes of the star at its upper and lower culminations. Since

$$h_1 = \varphi + p \quad \text{and} \quad h_2 = \varphi - p,$$

we have

$$\varphi = \tfrac{1}{2}(h_1 + h_2), \qquad p = \tfrac{1}{2}(h_1 - h_2).$$

Then, $r_1$ and $r_2$ denoting the probable errors of $h_1$ and $h_2$ respectively, that of $h_1 + h_2$ and also that of $h_1 - h_2$ is $\sqrt{(r_1^2 + r_2^2)}$, hence the probable error both of $\varphi$ and of $p$ when thus determined is

$$R = \tfrac{1}{2}\sqrt{(r_1^2 + r_2^2)}.$$

*The Linear Function of Several Observed Quantities.*

89. It follows from Arts. 84 and 87 that the linear function

$$Z = b + m_1 X_1 + m_2 X_2 + \ldots + m_n X_n \quad . \quad . \quad . \quad . \quad (1)$$

of $n$ observed quantities is subject to the usual law of facility,* its probable error being

$$R = \sqrt{(m_1^2 r_1^2 + m_2^2 r_2^2 + \ldots + m_n^2 r_n^2)}, \quad . \quad . \quad . \quad (2)$$

where $r_1, r_2, \ldots r_n$ are the probable errors of the several observed quantities.

In particular, if the $n$ quantities have the same probable error $r$, the probable error of their sum is $r\sqrt{n}$. The probable error of their arithmetical mean, which is $\frac{1}{n}$ of this sum, is therefore $\frac{r}{\sqrt{n}}$. This result agrees with that found in Art. 64, where,

* The fact that the law of facility thus reproduces itself has often been regarded as confirmatory of its truth. This property of the law $ce^{-h^2x^2}$ results from its being a limiting form for the facility of error in the linear function $Z$, when $n$ is large, whatever be the forms of the facility functions for $X_1, X_2, \ldots X_n$. Compare the foot-note on page 49, and see the memoir there referred to. It follows that "we shall obtain the same law $e^{-h^2x^2}$ (for a single observed quantity) if we regard each actual error as formed by the linear combination of a large number of errors due to different independent sources."

however, the $n$ quantities were all observed values of the same quantity, and the arithmetical mean was under consideration by virtue of its being the most probable value in accordance with the law of facility.

**90.** It is to be noticed that in formula (2) it is essential that the probable errors $r_1, r_2, \ldots r_n$ should be the results of independent determinations. For example, in the illustration given in Art. 88, we have $h_1 = \varphi + p$, whence we should expect to find

$$(\text{prob. err. of } h_1)^2 = (\text{prob. err. of } \varphi)^2 + (\text{prob. err. of } p)^2;$$

but it will be found that this is not true when the probable errors of $\varphi$ and of $p$ are determined as in that article. In fact, in the demonstration given in Art. 86, it is assumed that the law of facility for $Y$ holds true when $X$ has a definite fixed value; but in the present illustration the law of facility found for $\varphi$ does not hold true for a definite fixed value of $p$.*

*The Non-Linear Function of Several Observed Quantities.*

**91.** Supposing, as in Art. 85, that the errors of the observed quantities are small compared to the quantities themselves, we may replace any function by an approximately equivalent function of a linear form. Thus, denoting the true values of the observed quantities $X_1, X_2, \ldots X_n$ by $a_1, a_2, \ldots a_n$, we have

$$Z = f(X_1, X_2, \ldots X_n) = f(a_1 + x_1, a_2 + x_2, \ldots a_n + x_n).$$

Expanding, and neglecting powers and products of the small quantities $x_1, x_2, \ldots x_n$, we obtain the approximate value

$$Z = f(a_1, a_2, \ldots a_n) + x_1 \frac{df}{da_1} + x_2 \frac{df}{da_2} + \ldots + x_n \frac{df}{da_n},$$

which is of the linear form. Hence, in accordance with equation (2), Art. 89, the probable error of $Z$ may be determined by the equation

$$R^2 = r_1^2 \left(\frac{df}{dX_1}\right)^2 + r_2^2 \left(\frac{df}{dX_2}\right)^2 + \ldots + r_n^2 \left(\frac{df}{dX_n}\right)^2.$$

* If the value of $p$ were known, each value of $h_1$ would imply a special value of $h_2$, and therefore the probability of $\phi$ would no longer be that found in Art. 88.

*Examples.*

1. If the probable error in measuring the radius $a$ of a circle is $r$, what are the probable errors of the circumference and of the area? $2\pi r$; $2\pi ar$.

2. What is the probable error of $\log_{10} x$, $r$ being the probable error of $x$? $0.4343 \frac{r}{x}$.

3. If measurements of adjacent sides of a rectangle give $a \pm r_1$ and $b \pm r_2$, what is the probable error of the area $ab$?

$$\sqrt{(b^2 r_1^2 + a^2 r_2^2)}.$$

4. If the rectangle is found to be a square and the sides are measured with the same precision, show that the probable error of the area is the same as if it were known to be a square; but if $r_1$ and $r_2$ are not equal, the area is obtained with less accuracy than it would be if it were known to be a square.

5. An angle observation is the difference between two readings of the limb of the instrument; if $r$ is the probable error of the angle, what is the probable error of each reading? $\frac{r}{\sqrt{2}}$.

6. The zenith distance of a star observed in the meridian is

$$\zeta = 21^\circ\ 17'\ 20''.3, \text{ with the mean error } 2''.3,$$

and the declination of the star is given

$$\delta = 19^\circ\ 30'\ 14''.8, \text{ with the mean error } 0''.8:$$

what is the mean error of the latitude of the place of observation found from the formula $\varphi = \zeta + \delta$?

$$\varphi = 40^\circ\ 47'\ 35''.1, \text{ with the mean error } 2''.44.$$

7. The latitude of a place has been found with the mean error $0''.25$, and the meridian zenith distance of stars observed at that place with a certain instrument has been found to be subject to the mean error $0''.62$; what is the mean error of the declinations of the stars deduced by the formula $\delta = \varphi - \zeta$? $0''.67$.

8. The correction of a chronometer is found to be $+12^m\ 13^s.2$, with the mean error $0^s.3$; ten days later the correction is found to be $+12^m\ 21^s.4$, with the same mean error; what is the mean daily rate and its mean error? $+0^s.82$; $0^s.042$.

9. If the error of a single measurement of an angle by a repeating circle consists of parts due to sighting and reading respectively, so that

$$r^2 = r_1^2 + r_2^2,$$

show that the probable error when the angle is repeated $n$ times is

$$\sqrt{\left(\frac{r_1^2}{n} + \frac{r_2^2}{n^2}\right)}.$$

10. If the measured sides of a rectangle have the same probable error, show that the diagonal is determined with the same precision as either side.

11. The compression of the earth's meridian was found to be $\frac{1}{294}$, with a probable error of .000046; what is the probable error of the denominator 294? 3.98.

12. When a line whose length is $l$ is measured by the repeated application of a unit of measure, show that its probable error is of the form

$$R = r\sqrt{l}.$$

13. What is the probable error of the area of the rectangle whose sides measured as in the preceding example are $z_1$ and $z_2$?

$$r\sqrt{[z_1 z_2 (z_1 + z_2)]}.$$

14. A line of levels is run in the following manner: the back and fore sights are taken at distances of about 200 feet, so that there are thirteen stations per mile, and at each sight the rod is read three times. If the probable error of a single reading is 0.01 of a foot, what is the probable error of the difference of level of two points which are ten miles apart? .093.

15. Show that the probable error of the weighted mean of observed quantities has its least possible value when the weights are inversely proportional to the squares of the probable errors of the quantities, and that this value is the same as that given in Art. 68 for the case of observed value of the same quantity.

# VII.

## The Combination of Independent Determinations of the Same Quantity.

### *The Distinction between Precision and Accuracy.*

**92.** We have seen in Arts. 63 and 67 that the final determination of the observed quantity derived from a set of observations follows the exponential law of the facility of accidental errors. The discrepancies of the observations have given us the means of determining a measure of the risk of error in the single observations, and we have found that the like measure for the final determination varies inversely as the square root of its weight compared with that of the single observation. Since this weight increases directly with the number of constituent observations, it is thus possible to diminish the risk of error indefinitely; in other words, to increase without limit the *precision* of our final result.

**93.** It is important to notice, however, that this is by no means the same thing as to say that it is possible by multiplying the number of observations to increase without limit the *accuracy* of the result. The precision of a determination has to do only with the accidental errors; so that the diminution of the probable error, while it indicates the reduction of the risk of such errors, gives no indication of the *systematic** errors (see Art. 3)

* The term systematic is sometimes applied to errors produced by a cause operating in a systematic manner upon the several observations, thus producing discrepancies obviously not following the law of accidental errors. Usually a discussion of these errors leads to the discovery of their cause, and ultimately to the corrections by means of which they may be removed. All the remaining errors, whose causes are unknown, are generally spoken of as accidental errors; but in this book the term accidental is applied only to those errors which are variable in the system of observations under consideration, as distinguished from those which have a common value for the entire system.

which are produced by unknown causes affecting all the observations of the system to exactly the same exte[illegible] that [illegible]

The value to which we appro[illegible] kind shall diff[illegible]n of the determination is inc[illegible] $\times \sqrt{2}$ [illegible]n of as the "true value," but [illegible] more properly *the precise value* corresponding to the instrument or method of observation employed. Since the systematic error is common to the whole system of observations, it is evident that it will enter into the final result unchanged, no matter what may be the number of observations; whereas the object of increasing this number is to allow the accidental errors to destroy one another. Thus the systematic error is the difference between the precise value, from which accidental errors are supposed to be entirely eliminated, and the accurate or true value of the quantity sought.

**94.** Hence, when in Art. 64 the arithmetical mean of $n$ observations was compared to an observation made with a more precise instrument, it is important to notice that this new instrument must be imagined to lead to the same ultimate precise value, that is, it must have the same systematic error as the actual instrument, whereas in practice a new instrument might have a very different systematic error.

Again, in the illustration employed in Art. 64, where the final determination of an angle is given as $36° \, 42'.3 \pm 1'.22$, the "true value," which is just as likely as not to lie between the limits thus assigned, is only the true value so far as the instrument and method employed can give it; that is, the precise value to which the determination would approach if its weight were increased indefinitely.

**95.** A failure to appreciate the distinction drawn in the preceding articles may lead to a false estimate of the value of the method of Least Squares. M. Faye in his "Cours d'Astronomie" gives the following example of the objections which have been urged against the method: "From the discussion of the transits of Venus observed in 1761 and 1769, M. Encke deduced for the parallax of the sun the value

$$8''.57116 \pm 0''.0370.$$

In accordance with this small probable error it would be a wager of one to one that the true parallax is comprised between 8″ 61. Now we know to-day that the true parallax 8″.813 outside of these limits. The error, 0″.24184, is equal to 6.536 times the probable error 0″.037. We find for the probability of such an error 0.00001. Hence, adhering to the probable error assigned by M. Encke to his result, one could wager a hundred thousand to one that it is not in error by 0.24184, and nevertheless such is the correction which we are obliged to make it undergo."

Of course, as M. Faye remarks, astronomers can now point out many of the errors for which proper corrections were not made; but the important thing to notice is that, even in Encke's time, the wagers cited above were not authorized by the theory. The value of the parallax assigned by Encke was the most probable with the evidence then known, and it was an even wager that the complete elimination of errors of the kind that produced the discrepancies or contradictions among the observations could not carry the result beyond the limit assigned; but the existence of other unknown causes of error and the probable amount of inaccuracy resulting from them is quite a different question.

### *Relative Accidental and Systematic Errors.*

**96.** Let us now suppose that two determinations of a quantity have been made with the same instrument and by the same method, so that they have the same systematic error, if any; in other words, they correspond to the same precise value. The difference between the two results is the algebraic difference between the accidental errors remaining in the two determinations; this may be called their *relative accidental error.* Regarding the two determinations as independent measurements of two quantities, if $r_1$ and $r_2$ are their probable errors, that of their difference is $\sqrt{(r_1^2 + r_2^2)}$; and, since this difference should be zero, the relative error is an error in a system for which the probable error is

$$r = \sqrt{(r_1^2 + r_2^2)}.$$

For example, if the determination of an angle mentioned in Art. 94 is the mean of ten observations, it is an even wager that the mean of ten more observations of the same kind shall differ from 36° 42′.3 by an amount not exceeding 1′.22 × $\sqrt{2}$ or 1′.73. Again, $r$ being the probable error of a single observation, the probable error of the mean of $n$ observations is $\frac{r}{\sqrt{n}}$, but the discrepancy from this mean of a new single observation is as likely as not to exceed

$$\sqrt{\left(\frac{r^2}{n} + r^2\right)}, \quad \text{that is,} \quad r\sqrt{\frac{n+1}{n}}.*$$

**97.** If, on the other hand, the two determinations have been made with different instruments or by a different method, they may involve different systematic errors; so that, if each determination were made perfectly precise, they would still differ by an amount equal to the algebraic difference of their systematic errors. Let this difference, which may be called the *relative systematic error*, be denoted by $\delta$. Then, $d$ denoting the actual difference of the two determinations, while $\delta$ is the difference between the corresponding precise values, we may put

$$d = \delta + x,$$

in which $x$ is the relative accidental error.

*The Relative Weights of Independent Determinations.*

**98.** In combining values to obtain a final mean value, we have hitherto supposed their relative weights to be known or assumed beforehand, as in Arts. 75 and 77. Since the squares of the probable errors are inversely proportional to the weights, (Arts. 66 and 68,) the ratios of the probable errors both of the constituents and of the mean are thus known in advance, and it

* This does not apply to the residuals of the original $n$ observations, because in taking a residual the mean is not independent of the single observation with which it is compared.

only remains to determine a single absolute value of a probable error to fix them all. In this process it is assumed that the values have all the same systematic error.

But, when the determinations are independently made, their relative weights are not known, and their probable errors have to be found independently. If now it can be assumed that the systematic errors are the same, so that there is no relative systematic error, the weights may be taken in the inverse ratio of the squares of the probable errors.

**99.** To determine whether the above assumption can fairly be made in the case of two independent determinations whose probable errors are $r_1$ and $r_2$, it is necessary to compare the difference $d$ with the relative probable error $\sqrt{(r_1^2 + r_2^2)}$, Art. 96. If $d$ is small enough to be regarded as a relative accidental error, it is safe to make the assumption and combine the determinations in the manner mentioned above.

As an example, let us suppose that a certain angle has been determined by a theodolite as

$$24^\circ\ 13'\ 36'' \pm 3''.1,$$

and that a second determination made with a surveyor's transit is

$$24^\circ\ 13'\ 24'' \pm 13''.8.$$

In this case $r_1 = 3.1$, $r_2 = 13.8$ and $d = 12$. It is obvious that a relative accidental error as great as $d$ may reasonably be expected. (In fact the relative probable error is 14.1 ; and, by Table II, the chance that the accidental error should be at least as great as 12 is about .57.) We may therefore assume that there is no relative systematic error, and combine the determinations with weights having the inverse ratio of the squares of the probable errors. This ratio will be found, in the present case, to be about 20 : 1, and the corresponding weighted mean found by adding $\frac{1}{21}$ of the difference to the first value, is

$$24^\circ\ 13'\ 35''.43.$$

**100.** It appears doubtful at first that the value given by the

theodolite can be improved by combining with it the value given by the inferior instrument. The propriety of the above process becomes more apparent, however, if we imagine the first determination to be the mean of twenty observations made with the theodolite; a single one of these observations will then have the same weight and the same probable error as the second determination. Now the discrepancy of this new determination from the mean is such as we may expect to find in a new single observation with the theodolite. We are therefore justified in treating it as such an observation, and taking the mean of the twenty-one supposed observations for our final result.

**101.** The probable error of the result found in Art. 99 of course corresponds with its weight; thus, denoting it by $R$, we have $R^2 = \frac{20}{21} r_1^2$, whence $R = 3''.03$, and the final result is

$$24^\circ\ 13'\ 35''.43 \pm 3''.03.$$

In general, $r_1$ and $r_2$ being the given probable errors, that of the mean is given by

$$R^2 = \frac{r_1^2 r_2^2}{r_1^2 + r_2^2}.$$

Determinations which, considering their probable errors, are in sufficient agreement to be treated as in the foregoing articles may be called *concordant determinations*. They correspond to the same precise value of the observed quantity, and the result of their combination is to be regarded as a better determination of the same precise value.

### *The Combination of Discordant Determinations.*

**102.** As a second illustration of determinations independently made, let us suppose that a determination of the zenith distance of a star made at one culmination is

$$14^\circ\ 53'\ 12''.1 \pm 0''.3,$$

and that at another culmination we find for the same quantity

$$14^\circ\ 53'\ 14''.3 \pm 0''.5.$$

In this case we have $d = 2.2$. This is about 3.8 times the relative probable error whose value is $0''.58$.

From Table II we find that the probability that the relative accidental error should be as great as $d$ is only about 1 in 100. We are therefore justified in assuming that the difference $d$ is mainly due to errors peculiar to the culminations. In other words, we assume that, could we have obtained the precise values corresponding to the two culminations, (by indefinitely increasing the number of observations at each,) they would still be found to differ by about $2''.2$. Supposing now that there is no reason for preferring one of these precise values to the other, we ought to take their simple arithmetical mean for the final result; and, since the two given values are comparatively close to the precise values in question, we may take their arithmetical mean, which is

$$14^\circ\ 53'\ 13''.2,$$

for the final determination.

**103.** Determinations like those considered above, whose difference is so great as to indicate an actual difference between the precise values to which they tend, may be called *discordant determinations.* The discordance of the two determinations discloses the existence of systematic errors which were not indicated by the discrepancies of the observations upon which the given probable errors were based. In combining the determinations, these systematic errors are treated as accidental errors incident to the two determinations considered as two observed values of the required quantity. In fact, it is generally the object in making new and independent determinations to eliminate as far as possible a new class of errors by bringing them into the category of accidental errors which tend to neutralize each other in the final result. The probable error of the result cannot now be derived from the given probable errors, but must be inferred from the determinations themselves considered as observed values, because we now take cognizance of errors which are not indicated by the given probable errors.

**104.** When there are but two observed values, formula (4), Art. 72, becomes

$$R_0 = \rho\sqrt{2}\sqrt{\frac{p_1 v_1^2 + p_2 v_2^2}{p_1 + p_2}},$$

in which $p_1$, $p_2$ are the weights assigned to the two values. Denoting the difference by $d$, the residuals have opposite signs, and their absolute values are

$$v_1 = \frac{p_2}{p_1 + p_2} d, \qquad v_2 = \frac{p_1}{p_1 + p_2} d.$$

Substituting these values, we have for the probable error of the mean

$$R_0 = \rho \frac{\sqrt{(2p_1p_2)}}{p_1 + p_2} d \; 0.6745 \frac{\sqrt{(p_1p_2)}}{p_1 + p_2} d. \quad . \quad . \quad . \quad (1)$$

When $p_1 = p_2$, this becomes

$$R_0 = \frac{\rho d}{\sqrt{2}} = 0.3372\, d. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the example given in Art. 102, the value of $R_0$ thus obtained is $0''.742$, which, owing to the discordance of the two given determinations, considerably exceeds each of the given probable errors.

Of course no great confidence can be placed in the results given by the formulæ above on account of the small value of $n$.*

**105.** Since the error of each determination is the sum of its accidental and systematic error, if $s_1$ and $s_2$ denote the probable

* The argument by which it is shown that the value of $h$ deduced in Art. 69 is the most probable value involves the assumption that before the observations were made all values of $h$ are to be regarded as equally probable; just as that by which it is shown that the arithmetical mean is the most probable value of the observed quantity $a$ involves the assumption that before the observations all values of $a$ were equally probable. In the case of $a$, the assumption is admissible with respect to all values of $a$ which can possibly come in question. But, in the case of $h$, this is not true; because (supposing $n = 2$ as above) when $d = 0$ the value of $h$ is infinite, and when $d$ is small the corresponding values of $h$ are very large, so that it is impossible to admit that all values of $h$ which can arise are *à priori* equally probable.

In the present application of the formula, however, these inadmissible values do not arise, because we do not use it when $d$ is small, employing instead the method of Art. 99 and the formula of Art. 101.

systematic errors, the probable errors of the two determinations when both classes of errors are considered are

$$R_1 = \sqrt{(r_1^2 + s_1^2)}, \qquad R_2 = \sqrt{(r_2^2 + s_2^2)}.$$

The proper ratio of weights with which the determinations should be combined is $R_2^2 : R_1^2$. The method of procedure followed in Art. 99 assumes that $s_1$ and $s_2$ vanish. On the other hand, in the process employed in Art. 102 we are guided, in an assumption of the ratio $R_2^2 : R_1^2$, by a consideration of the value which the ratio $s_2^2 : s_1^2$ ought to have.

For example, in the illustration, Art. 102, the ratio $R_2^2 : R_1^2$ is taken to be one of equality, whereas the hypothesis we desired to make was that $s_1 = s_2$, so that we ought to have

$$R_1^2 - R_2^2 = r_1^2 - r_2^2.$$

On the hypothesis $R_1 = R_2$ the value of each of these probable errors is, in accordance with equation (2), Art. 104, $\rho d$. In the example this is 1″.05. If we take $(1.05)^2$ as the average value of $R_1^2$ and $R_2^2$, and introduce the condition written above, we shall find as a second approximation to the value of the ratio $R_2^2 : R_1^2$ about 15 : 13. The final value corresponding to this ratio of weights is 14° 53′ 13″.1, and its probable error as determined by equation (1), Art. 104, is slightly less than that before found, namely, $R_0 = 0''.740$.

### *Indicated and Concealed Portions of the Risk of Error.*

**106.** It will be convenient in the following articles to speak of the square of the probable error as the measure of the risk of error.

The foregoing discussion shows that the total risk of error, $R^2$, of any determination consists of two parts, $r^2$ and $s^2$, of which the first only is indicated by discrepancies among the observations of which the given determination is the mean. It is only this first part that can be diminished by increasing the number of the constituent observations. The remaining part remains concealed, and cannot be diminished until some varia-

tion is made in the circumstances under which the observations are made, giving rise to new determinations. When the indicated portions of the risk of error in the several determinations are sufficiently diminished, discordance between them must always be expected, and this discordance brings into evidence a new portion, but still it may be only a portion, of the hitherto concealed part of the risk of error.

**107.** What we have called in Art. 103 discordant determinations are those in which the indication of this new portion of the risk of error, to which corresponds the relative systematic error, is unmistakable, because of its magnitude in comparison with what remains of the portion first indicated in the separate determinations, that is, $r_1^2$ and $r_2^2$. On the other hand, the concordant determinations of Art. 101 are those in which the new portion is so small compared with $r_1^2$ and $r_2^2$ as to remain concealed.

Thus, to return to the illustration discussed in Art. 99, if twenty times as many observations had been involved in the determination by the transit, its probable error would have been reduced to equality with that of the determination by the theodolite. But if this had been done we should almost certainly have found the determinations discordant; that is to say, the ratio in which the difference between the determinations is reduced would be much less than that in which the probable relative accidental error $\sqrt{(r_1^2 + r_2^2)}$ is diminished. The ratio in which the remaining difference between the determinations should be divided in making the final determination now depends upon our estimate of the comparative freedom of the instruments from systematic error,* but the important thing to be noted is that the probable error of the result would now be found as in Art. 104, and would be greater than those of the

*It may be assumed that, when the instruments are carefully adjusted, the one which is less liable to accidental errors is correspondingly less liable to systematic errors. But this comparison is concerned with the probable errors *of a single observation* in each case, and not with those of the determinations themselves.

separate determinations. Thus the *apparent* risk of error would be increased by making a new determination, but this is only because a greater part of the total risk of error has been made apparent, and the result is so much the more trustworthy as a greater variety has been introduced into the methods employed.

*The Total Probable Error of a Determination.*

**108.** In the illustrations given in Arts. 99 and 102 it was supposed that two determinations only were made, so that we had but a single discrepancy upon which to base our judgment of the probable amount of the relative systematic error. But, in general, what are regarded as determinations at one stage of the process are at the next stage treated as observations which may be repeated indefinitely before being combined into a new determination. Let one of the determinations first made be the mean of $n$ observations equally good, and let $r$ be the probable error of a single observation. Then the probable accidental error of the mean is $r_0 = \frac{r}{\sqrt{n}}$. Now, if $R$ is the probable error of the final value as obtained directly from the discrepancies of the several determinations, (their number being supposed great enough to allow us to obtain a trustworthy value,) we shall find that $R$ exceeds $r_0$, and putting

$$R^2 = \frac{r^2}{n} + r_1^2, \quad \ldots\ldots\ldots \quad (1)$$

$r_1^2$ is the new portion of the risk of error brought out by the comparison of the determinations.

**109.** The form of this equation shows that when $\frac{r^2}{n}$ is already small compared with $r_1^2$, the advantage gained by increasing the value of $n$ soon becomes inappreciable.

For example, the reticule of a meridian circle is provided with a number of threads, in order that several observations of time may be taken at a single transit. If seven equidistant threads are used, the mean of the times is equivalent to a determination

based upon seven observations of the time of transit. Chauvenet found that, for moderately skilful observers, the probable accidental error of the transit over a single thread of an equatorial star is $r = 0^s.08$, whence for the mean of the seven threads we have $r_0 = 0^s.03$. The probable error of a single determination of the right ascension of an equatorial star was found to be $R = 0^s.06$, so that, from $R^2 = r_0^2 + r_1^2$ we have $r_1 = 0^s.052$. The conclusion is reached that "an increase of the number of threads would be attended by no important advantage," and it is stated that Bessel thought five threads sufficient.*

**110.** Suppose the value of $R^2$ in equation (1), Art. 108, to have been derived from the discrepancies of $n'$ determinations of equal weight. A systematic error may exist for these $n'$ determinations, and $s_1$ being its probable value, we shall have

$$s^2 = r_1^2 + s_1^2,$$

that is to say, the concealed portion of the risk of error in one of the original determinations has been decomposed into two parts, one of which has been disclosed at the second stage of the process, while the other remains concealed.

The total risk of error in a single one of the $n'$ determinations is $R^2 + s_1^2$, and that of the mean of the determinations is $\frac{R^2}{n'} + s_1^2$.

In like manner, if at a further stage of the process we have the means of finding the value of the probable error $R_1$ of this new determination by direct comparison with other coordinate determinations, a portion of the value of $s_1^2$ will be disclosed, and we shall have

$$R_1^2 = \frac{R^2}{n'} + r_2^2 = \frac{r^2}{nn'} + \frac{r_1^2}{n'} + r_2^2,$$

where again it must be supposed that a portion $s_2^2$ of the risk of error still remains concealed.

* Chauvenet's "Spherical and Practical Astronomy," vol. ii, p. 194 et seq.

**111.** The comparative amounts of the risk of error which are disclosed at the various stages of the process depend upon the amount of variety introduced into the method of observing. Thus, to resume the illustration given in Art. 109, if the star be observed at $n'$ culminations, $r^2$ will correspond to errors peculiar to a thread, and $r_1^2$ will correspond to errors peculiar to a culmination. Again, if different stars whose right ascensions are known are observed, in order to obtain the local sidereal time used in a determination of the longitude, $r_2^2$ will correspond to errors peculiar to a star, together with instrumental errors peculiar to the meridian altitude.

### *The Ultimate Limit of Accuracy.*

**112.** The considerations adduced in the preceding articles seem to point to the conclusion that there must always be a residuum of the risk of error that has not yet been reached, and thus to explain the apparent existence "of an ultimate limit of accuracy beyond which no mass of accumulated observations can ever penetrate."* But it does not appear to be necessary to suppose, as done by Professor Peirce, that there is an absolute fixed limit of accuracy, due to "a failure of the law of error embodied in the method of Least Squares, when it is extended to minute errors." He says: "In approaching the ultimate limit of accuracy, the probable error ceases to diminish proportionally to the increase of the number of observations, so that the accuracy of the mean of several determinations does not surpass that of the single determinations as much as it should do, in conformity with the law of least squares; thus it appears that the probable error of the mean of the determinations of the longitude of the Harvard Observatory, deduced from the moon-culminating observations of 1845, 1846, and 1847, is $1^s.28$ instead of $1^s.00$, to which it should have been reduced conformably to the accuracy of the separate determinations of those years."

* Prof. Benjamin Peirce, *U. S. Coast Survey Report* for 1854, Appendix, p. 109.

To account for the fact cited on the principles laid down above, it is only necessary to suppose that there are causes of error which have varied from year to year; and, recognizing this fact, we ought to obtain our final determination by comparing the determinations of a number of years, and not by combining into one result the whole mass of observations.

### *Examples.*

1. In a system of observations equally good, $r$ being the probable error of a single observation, if two observations are selected at random, what quantity is their difference as likely as not to exceed? $r\sqrt{2}$.

2. In example 1, what is the probability that the difference shall be less than $r$? 0.367.

3. When two determinations are made by the same method, show that the odds are in favor of a difference less than the sum of the two probable errors, and against a difference less than the greater of the two, and find the extreme values of these odds.

66 : 34 and 63 : 37.

4. A and B observe the same angle repeatedly with the same instrument, with the following results:

| A | B |
|---|---|
| 47° 23′ 40″ | 47° 23′ 30″ |
| 47 23 45 | 47 23 40 |
| 47 23 30 | 47 23 50 |
| 47 23 35 | 47 24 00 |
| 47 23 40 | 47 23 20 |

Show that there is no evidence of relative systematic (personal) error. Find the relative weights of an observation by A and by B, and the final determination of the angle.

100 : 13; 47° 23′ 38″.23 ± 1″.62.

5. Show that the probable error in example 4 as computed from the ten observations taken with their proper weights is 1″.53, but that derived from the formula of Art. 104 is 0″.43, which is much too small. (See foot-note, p. 83.)

6. Two determinations of the length of a line in feet give respectively $683.4 \pm 0.3$ and $684.9 \pm 0.3$, there being no reason for preferring one of the corresponding precise values to the other; show that the probable error of each of the precise values (that is, the systematic error of each determination) is 0.65; and that the best final determination is $684.15 \pm 0.51$.

7. Show generally that when the weights are inversely proportional to the squares of the probable errors, the formula of Art. 104 gives a value of $R$ greater or less than that given by the formula of Art. 101, according as $d$ is greater or less than the relative mean error.

# VIII.

## Indirect Observations.

### *Observation Equations.*

**113.** We have considered the case in which a quantity whose value is to be determined is directly observed, or is expressed as a function of quantities directly observed. We come now to that in which the quantity sought is one of a number of unknown quantities of which those directly observed are functions. The equation expressing that a known function of several unknown quantities has a certain observed value is called an *observation equation.* Let $\mu$ denote the number of unknown quantities concerned. Then, in order to determine them, we must have at least $\mu$ independent equations. Thus, if two of the equations express observed values of the same function of the unknown quantities, they will either be identical, so that we have in effect only $\mu - 1$ equations, or else they will be inconsistent, so that the values of the unknown quantities will be impossible. So also it must not be possible to derive any one of the $\mu$ equations, or one differing from it only in the absolute term, from two or more of the other equations.

**114.** If we have no more than the necessary $\mu$ equations, we shall have no indication of the precision with which the observations have been made, nor, consequently, any measure of the precision with which the unknown quantities have been determined. With respect to them, we are in the same condition as when a single observed value is given in the case of direct observations.

Now let other observation equations be given, that is to say, let the values of other functions* of the unknown quantities be observed. The results of substituting the values of the unknown

* It is not necessary that these additional equations should be independent of the original $\mu$ equations, for an equation expressing a new observed value of a function already observed will be useful in determining the precision of the observations.

quantities will, owing to the errors of observation, be found to differ from the observed values, and the discrepancies will give an indication of the precision of the observations, just as the discrepancies between observed values of the same quantity do, in the case of direct observations.

**115.** As an example, let us take the following four observation equations* involving $x, y$ and $z$:

$$\begin{aligned} x - y + 2z &= 3, \\ 3x + 2y - 5z &= 5, \\ 4x + y + 4z &= 21, \\ -x + 3y + 3z &= 14. \end{aligned}$$

If we solve the first three equations we shall find

$$x = 2\tfrac{4}{7}, \qquad y = 3\tfrac{2}{7}, \qquad z = 1\tfrac{6}{7}.$$

Substituting these values in the fourth equation, the value of the first member is $12\frac{6}{7}$, whereas the observed value is 14; the discrepancy is $1\frac{1}{7}$. If the values above were the true values, the errors of observation committed must have been 0, 0, 0, $1\frac{1}{7}$; but, since each of the observed quantities is liable to error, this is not a likely system of errors to have been committed. In fact, any system of values we may assign to $x, y$ and $z$ implies a system of errors in the observed quantities, and the most probable system of values is that to which corresponds the most probable system of errors.

**116.** In general, let there be $m$ observation equations, involving $\mu$ unknown quantities, $m > \mu$; then we have first to consider the mode of deriving from them the most probable values of the unknown quantities. The system of errors in the observed quantities which this system of values implies will then enable us to measure the precision of the observations. Finally, regarding the $\mu$ unknown quantities as functions of the $m$ observed quantities, we shall obtain for each unknown quantity a measure of the precision with which it has been determined.

* Gauss, "Theoria Motus Corporum Coelestium," Art. 184.

*The Reduction of Observation Equations to the Linear Form.*

**117.** The method of obtaining the values of the unknown quantities, to which we proceed, requires that the observation equations should be linear. When this is not the case, it is necessary to employ approximately equivalent linear equations, which are obtained in the following manner.

Let $X, Y, Z, \ldots$ be the unknown quantities, and $M_1, M_2, \ldots M_m$ the observed quantities; the observation equations are then of the form

$$\begin{aligned} f_1(X, Y, Z, \ldots) &= M_1, \\ f_2(X, Y, Z, \ldots) &= M_2, \\ &\cdots\cdots\cdots \\ f_m(X, Y, Z, \ldots) &= M_m, \end{aligned}$$

where $f_1, f_2, \ldots f_m$ are known functions. Let $X_0, Y_0, Z_0, \ldots$ be approximate values of $X, Y, Z, \ldots$, which, if not otherwise known, may be found by solving $\mu$ of the equations; and put

$$X = X_0 + x, \qquad Y = Y_0 + y, \quad \ldots,$$

so that $x, y, z, \ldots$ are small corrections to be applied to the approximate values. Then the first observation equation may be written

$$f_1(X_0 + x, Y_0 + y, Z_0 + z, \ldots) = M_1,$$

or, expanding by Taylor's theorem,

$$f_1(X_0, Y_0, Z_0, \ldots) + \frac{df_1}{dX_0}x + \frac{df_1}{dY_0}y + \frac{df_1}{dZ_0}z + \ldots = M_1,$$

where the coefficients of $x, y, z, \ldots$ are the values which the partial derivatives of $f_1(X, Y, Z, \ldots)$ assume when $X = X_0$, $Y = Y_0$, $Z = Z_0$, $\ldots$, and the powers and products of the small quantities $x, y, z, \ldots$ are neglected as in Art. 91.

Denoting the coefficients of $x, y, z, \ldots$ by $a_1, b_1, c_1, \ldots$, putting $n_1$ for $M_1 - f_1(X_0, Y_0, Z_0, \ldots)$, and treating the other observation equations in the same way, we may write

$$\left.\begin{array}{l} a_1x + b_1y + c_1z + \ldots = n_1 \\ a_2x + b_2y + c_2z + \ldots = n_2 \\ \cdots\cdots\cdots\cdots \\ a_mx + b_my + c_mz + \ldots = n_m \end{array}\right\}, \quad . \quad . \quad . \quad (1)$$

for the observation equations in their linear form.

**118.** Even when the original observation equations are in the linear form, it is generally best to transform them as above, so that the values of the unknown quantities shall be small.

Another transformation sometimes made consists in replacing one of the unknown quantities by a fixed multiple of it. For example, if the values of the coefficients of $y$ are inconveniently large they may be reduced in value by substituting $ky'$ for $y$ and giving to $k$ a suitably small value.

**119.** In the observation equations (1), the second members may be regarded as the observed quantities, since they have the same errors. If the true values of $x, y, z, \ldots$ are substituted in these equations they will not be satisfied, because each $n$ differs from its proper value by the error of observation $v$; we may therefore write the equations

$$\left.\begin{array}{l} a_1x + b_1y + c_1z + \ldots - n_1 = v_1 \\ a_2x + b_2y + c_2z + \ldots - n_2 = v_2 \\ \cdots\cdots\cdots\cdots \\ a_mx + b_my + c_mz + \ldots - n_m = v_m \end{array}\right\}, \quad . \quad . \quad . \quad (2)$$

in which, if $x, y, z, \ldots$ are the true values, $v_1, v_2, \ldots v_m$ are the true errors of observation, and if any set of values be given to $x, y, z, \ldots$, the second members are the corresponding *residuals*. These corrected observation equations may be called the *residual equations*.

## *Observation Equations of Equal Precision.*

**120.** Let us first suppose that the $m$ observations are equally good, and let $h$ be their common measure of precision. Then, since $v$ is the error, not only of the absolute term $n_1$ in the first of equations (2), but of the first observed quantity $M_1$, the prob-

ability before the observations are made that the first observed value shall be $M_1$ is

$$\frac{h}{\sqrt{\pi}}\, e^{-h^2 v_1^2} \Delta v,$$

where, as in Art. 35, $\Delta v$ is the least count of the instrument. Hence we have, for the probability before the observations are made that the $m$ actual observed values shall occur,

$$P = \frac{h^m}{\pi^{\frac{1}{2}m}}\, e^{-h^2(v_1^2 + v_2^2 + \cdots v_m^2)} \Delta v^m,$$

exactly as in Art. 41. The values of $v_1^2, v_2^2, \ldots v_m^2$ being given by equations (2), this value of $P$ is a function of the several unknown quantities; hence it follows, as in Art. 41, that for any one of them that value is, after the observations have been made, most probable which assigns to $P$ its maximum value; in other words, that value which makes

$$v_1^2 + v_2^2 + \ldots + v_m^2 = \text{a minimum.}$$

Thus the principle of Least Squares applies to indirect as well as to direct observations.

**121.** To determine the most probable value of $x$, we have, by differentiation with respect to $x$,

$$v_1 \frac{dv_1}{dx} + v_2 \frac{dv_2}{dx} + \ldots + v_m \frac{dv_m}{dx} = 0,$$

or, since, from equations (2), Art. 119,

$$\frac{dv_1}{dx} = a_1, \quad \frac{dv_2}{dx} = a_2, \quad \ldots \quad \frac{dv_n}{dx} = a_n,$$

$$a_1 v_1 + a_2 v_2 + \ldots + a_m v_m = 0. \quad . \quad . \quad . \quad . \quad (1)$$

This is called the *normal equation for* $x$. Whatever values are assigned to $y, z, \ldots$, it gives the rule for determining the value of $x$ which is most probable on the hypothesis that the values assigned to the other unknown quantities are correct.

Since $v_1, v_2, \ldots v_m$ represent the first members of the obser-

vation equations (1), Art. 117, when so written that the second member is zero, we see that *the normal equation for $x$ may be formed by multiplying each observation equation by the coefficient of $x$ in it, and adding the results.*

**122.** The rule just given for forming the normal equation shows it to be a linear combination of the observation equations, and the reason why the multipliers should be as stated may be further explained as follows: If we suppose fixed values given to $y, z, \ldots$, each observation equation may be written in the form $ax = N$, where $N$ only differs from the observed value $M$ by a fixed quantity, and therefore has the same probable error. Now, writing the observation equations in the form

$$x = \frac{N_1}{a_1} = x_1,$$

$$x = \frac{N_2}{a_2} = x_2,$$

$$\ldots\ldots\ldots$$

$$x = \frac{N_m}{a_m} = x_m,$$

we may regard them as expressing direct observations of $x$. If $r$ is the common probable error of $N_1, N_2, \ldots N_m$, that of $\frac{N_1}{a_1}$ or $x_1$ is $\frac{r}{a_1}$; that of $x_2$ is $\frac{r}{a_2}$, and so on. Thus the equations are not of equal precision for determining $x$, and their weights when written as above (being inversely as the squares of the probable errors) are as $a_1^2 : a_2^2 : \ldots : a_m^2$. It follows that the equation for finding $x$ is, as in the case of the weighted arithmetical mean (see Art. 66), the result of adding the above equations multiplied respectively by $a_1^2, a_2^2, \ldots a_m^2$;* that is to say, it is the result of adding the original observation equations of the form $ax - N = 0$ multiplied respectively by $a_1, a_2, \ldots a_m$.

*It must not be assumed that the weight of the value of $x$, determined from the several normal equations, is $\Sigma a^2$, that of an observation being unity. This is its weight only upon the supposition that the absolute values of the other quantities are known.

*The Normal Equations.*

**123.** In like manner, for each of the other unknown quantities we can form a normal equation, and we thus have a system of equations whose number is equal to that of the unknown quantities. The solution of this system of normal equations gives the most probable values of the unknown quantities. Let us take for example the four observation equations given in Art. 115. Forming the normal equations by the rule given above, we have

$$\begin{aligned} 27x + 6y &= 88, \\ 6x + 15y + z &= 70, \\ y + 54z &= 107. \end{aligned}$$

The solution of this system of equations gives for the most probable values,

$$x = \frac{49154}{19899} = 2.47,$$

$$y = \frac{2617}{737} = 3.55,$$

$$z = \frac{12707}{6633} = 1.92.$$

**124.** Writing the observation equations in their general form,

$$\left.\begin{aligned} a_1x + b_1y + \ldots + l_1t &= n_1 \\ a_2x + b_2y + \ldots + l_2t &= n_2 \\ \cdots\cdots\cdots\cdots & \\ a_mx + b_my + \ldots + l_mt &= n_m \end{aligned}\right\}, \quad \ldots \quad (1)$$

we obtain for the normal equations in their general form,

$$\left.\begin{aligned} \Sigma a^2 . x + \Sigma ab . y + \ldots + \Sigma al . t &= \Sigma an \\ \Sigma ab . x + \Sigma b^2 . y + \ldots + \Sigma bl . t &= \Sigma bn \\ \cdots\cdots\cdots\cdots & \\ \Sigma al . x + \Sigma bl . y + \ldots + \Sigma l^2 . t &= \Sigma ln \end{aligned}\right\} \ldots \quad (2)$$

It will be noticed that the coefficient of the $r$th unknown quantity in the $s$th equation is the same as that of the $s$th unknown quantity in the $r$th equation; in other words, the

determinant of the coefficients of the unknown quantities in equations (2) is a symmetrical one.

*Observation Equations of Unequal Precision.*

**125.** When the observations are not equally good, if

$$h_1, h_2, \ldots h_m$$

are the measures of precision of the observed values

$$M_1, M_2, \ldots M_m,$$

the expression to be made a minimum is

$$h_1^2 v_1^2 + h_2^2 v_2^2 + \ldots + h_m^2 v_m^2,$$

as in Art. 65. Thus, as in the case of direct observations, if the error of each observation be multiplied by its measure of precision so as to reduce the errors to the same relative value, *it is necessary that the sum of the squares of the reduced errors should be a minimum.*

Since $v_1 = 0, v_2 = 0, \ldots v_m = 0$ are equivalent to the observation equations, it follows that, if we multiply each observation equation by its measure of precision (so that it takes the form $hv = 0$), we may regard the results as equations of equal precision.

**126.** The result may be otherwise expressed by using numbers $p_1, p_2, \ldots p_m$ proportional, as in Art. 66, to the squares of the measures of precision; the quantity to be made a minimum then is

$$p_1 v_1^2 + p_2 v_2^2 + \ldots + p_m v_m^2,$$

and the normal equation for $x$ is

$$p_1 a_1 v_1 + p_2 a_2 v_2 + \ldots + p_m a_m v_m = 0.$$

The numbers $p_1, p_2, \ldots p_m$ are called the weights of the observation equations; thus, in the case of weighted equations, *the normal equation for x may be formed by multiplying each observation equation by the coefficient of x in it, and also by its weight, and adding the results.*

The general form of the normal equations is now

$$\left.\begin{array}{l} \Sigma pa^2 \,.\, x + \Sigma pab \,.\, y + \ldots + \Sigma pal \,.\, t = \Sigma pan \\ \Sigma pab \,.\, x + \Sigma pb^2 \,.\, y + \ldots + \Sigma pbl \,.\, t = \Sigma pbn \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \Sigma pal \,.\, x + \Sigma pbl \,.\, y + \ldots + \Sigma pl^2 \,.\, t = \Sigma pln \end{array}\right\} \quad . \quad . \quad (3)$$

The result is evidently the same as if each observation equation had been first multiplied by the square root of its weight, by which means it would be reduced to the weight unity, and the system would take the form (2), Art. 124.

### *Formation of the Normal Equations.*

**127.** When the normal equations are calculated by means of their general form, a table of squares is useful not only in calculating the coefficients $\Sigma pa^2$, $\Sigma pb^2$, . . . $\Sigma pl^2$, but also in the case of those of the form $\Sigma pab$, $\Sigma pac$, . . . $\Sigma pan$, . . . For, since

$$ab = \tfrac{1}{2}[(a+b)^2 - a^2 - b^2],$$

we have

$$\Sigma pab = \tfrac{1}{2}[\Sigma p(a+b)^2 - \Sigma pa^2 - \Sigma pb^2],$$

by means of which $\Sigma pab$ is expressed in terms of squares.* Or for the same purpose we may use

$$\Sigma pab = \tfrac{1}{2}[\Sigma pa^2 + \Sigma pb^2 - \Sigma p(a-b)^2].$$

In performing the work it is convenient to arrange the coefficients in a tabular form in the order in which they occur in the observation equations, and, adding a column containing the sums of the coefficients in each equation, thus,

$$s_1 = a_1 + b_1 + \ldots + l_1 + n_1, \quad \text{etc.},$$

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* If $\Sigma pab$ alone were to be found, the formula

$$\Sigma pab = \tfrac{1}{4}[\Sigma p(a+b)^2 - \Sigma p(a-b)^2],$$

derived from that of quarter-squares, would be preferable; but, since $\Sigma pa^2$, $\Sigma pb^2$ have also to be calculated, the use of the formula above, which was suggested by Bessel, involves less additional labor.

to form the quantities $\Sigma pas$, $\Sigma pbs$, ... $\Sigma pns$ in addition to those which occur in the normal equations. We ought then to find

$$\begin{aligned}
\Sigma pas &= \Sigma pa^2 + \Sigma pab + \ldots + \Sigma pan, \\
\Sigma pbs &= \Sigma pab + \Sigma pb^2 + \ldots + \Sigma pbn, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\Sigma pns &= \Sigma pan + \Sigma pbn + \ldots + \Sigma pn^2,
\end{aligned}$$

and the fulfilment of these conditions is a verification of the accuracy of the work.

In many cases, the use of logarithms is to be preferred, especially when the logarithms of the coefficients in the observation equations are more readily obtained than the values themselves.

*The General Expressions for the Unknown Quantities.*

**128.** In writing general expressions for the most probable values of the unknown quantities, and in deriving their probable errors, we shall, for simplicity in notation, suppose that the observation equations have been reduced to the weight unity as explained in Art. 126, so that they are represented by equations (1), and the normal equations by equations (2) of Art. 124.

Let $D$ be the symmetrical determinant of the coefficients of the unknown quantities in the normal equations, thus

$$D = \begin{vmatrix} \Sigma a^2 & \Sigma ab & \ldots & \Sigma al \\ \Sigma ab & \Sigma b^2 & \ldots & \Sigma bl \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \Sigma al & \Sigma bl & \ldots & \Sigma l^2 \end{vmatrix};$$

let $D_x$ denote the result of replacing the first column by a column consisting of the second members, $\Sigma an$, $\Sigma bn$, ... $\Sigma ln$; and let $D_y$, $D_z$, ... $D_t$ be the like results for the remaining columns. Then

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad \ldots \quad t = \frac{D_t}{D} \quad . \quad . \quad (1)$$

are the general expressions for the unknown quantities.

**129.** Let the value of $x$ when expanded in terms of the second members of the normal equations be

$$x = Q_1 \Sigma an + Q_2 \Sigma bn + \ldots + Q_\mu \Sigma ln. \quad . \quad . \quad (2)$$

Now, in the expansion of the determinant $D_x$ in terms of the elements of its first column, the coefficients of $\Sigma an$, $\Sigma bn, \ldots \Sigma ln$ are the first minors corresponding to $\Sigma a^2$, $\Sigma ab, \ldots \Sigma al$, in the determinant $D$.

Denoting the first of these by $D_1$, so that

$$D_1 = \begin{vmatrix} \Sigma b^2 & \Sigma bc & \ldots & \Sigma bl \\ \Sigma bc & \Sigma c^2 & \ldots & \Sigma cl \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \Sigma bl & \Sigma cl & \ldots & \Sigma l^2 \end{vmatrix},$$

it follows, on comparing the values of $x$ in equations (1) and (2), that

$$Q_1 = \frac{D_1}{D}.$$

In like manner, the values of $Q_2$, $Q_3, \ldots Q_\mu$ are the results of dividing the other first minors by $D$.

*The Weights of the Unknown Quantities.*

**130.** Let the value of $x$, when fully expanded in terms of the second members $n_1, n_2, \ldots n_m$ of the observation equations, be

$$x = a_1 n_1 + a_2 n_2 + \ldots + a_m n_m. \quad . \quad . \quad . \quad . \quad (3)$$

Then, if $r_x$ denotes the probable error of $x$, and $r$ that of a standard observation, that is, the common probable error of each of the observed values $n_1, n_2, \ldots n_m$, we shall have, by Art. 89,

$$r_x^2 = r^2 . \Sigma a^2.$$

The precision with which $x$ has been determined is usually expressed by means of its weight, that of a standard observation

being taken as unity. The weights being inversely proportional to the squares of the probable errors, we have, therefore, for that of $x$,

$$p_x = \frac{1}{\Sigma a^2}.$$

**131.** Since the value of $x$ is obtained from the normal equations, we do not actually find the values of the $a$'s; we therefore proceed to express $\Sigma a^2$ in terms of the quantities which occur in the normal equations.

Equating the coefficients of $n_1, n_2, \ldots n_m$ in equations (2) and (3), we find

$$\left.\begin{aligned} a_1 &= a_1 Q_1 + b_1 Q_2 + \ldots + l_1 Q_\mu \\ a_2 &= a_2 Q_1 + b_2 Q_2 + \ldots + l_2 Q_\mu \\ &\cdots\cdots\cdots\cdots \\ a_m &= a_m Q_1 + b_m Q_2 + \ldots + l_m Q_\mu \end{aligned}\right\} \quad \ldots\ldots (1)$$

Multiplying the first of these equations by $a_1$, the second by $a_2$, and so on, and adding the results, we have

$$\Sigma a^2 = \Sigma aa \,.\, Q_1 + \Sigma ba \,.\, Q_2 + \ldots + \Sigma la \,.\, Q_\mu. \quad . \quad (2)$$

The value of $\Sigma aa$ is found by multiplying the first of equations (1) by $a_1$, the second by $a_2$, and so on, and adding. The result is

$$\Sigma aa = \Sigma a^2 \,.\, Q_1 + \Sigma ab \,.\, Q_2 + \ldots + \Sigma al \,.\, Q_\mu. \quad . \quad (3)$$

Multiplying this equation by $D$, the second member becomes the expansion of the determinant $D$ in terms of the elements of its first column. Hence

$$\Sigma aa = 1. \quad \ldots\ldots\ldots (4)$$

In like manner we find

$$\Sigma ba = \Sigma ab \,.\, Q_1 + \Sigma b^2 \,.\, Q_2 + \ldots + \Sigma bl \,.\, Q_\mu, \quad . \quad (5)$$

and when this equation is multiplied by $D$, the second member is the expansion of a determinant in which the first two columns

are identical. Thus $\Sigma ba = 0$, and in the same way we can show that $\Sigma ca, \ldots \Sigma la$ vanish.*

Substituting in equation (2), we have now

$$\Sigma a^2 = Q_1; \quad \ldots \ldots \ldots \quad (6)$$

hence from Arts. 130 and 129 we have, for the general expression for the weight of $x$,

$$p_x = \frac{1}{Q_1} = \frac{D}{D_1}. \quad \ldots \ldots \ldots \quad (7)$$

**132.** It follows from equation (2), Art. 129, that if in solving the normal equations we retain the second members in algebraic form, putting for them $A, B, C, \ldots$, then *the weight of $x$ will be the reciprocal of the coefficient of $A$ in the value of $x$.*† In like manner, that of $y$ will be the reciprocal of the coefficient of $B$ in the value of $y$, and so on.

For example, if the normal equations given in Art. 123 are written in the form

$$\begin{aligned} 27x + 6y &= A, \\ 6x + 15y + z &= B, \\ y + 54z &= C, \end{aligned}$$

the solution is

$$\begin{aligned} 19899x &= 809A - 324B + 6C, \\ 737y &= -12A + 54B - C, \\ 6633z &= 2A - 9B + 123C. \end{aligned}$$

*Comparing equation (3) with equation (2), Art. 129, we see that $\Sigma aa$ is the value which $x$ would assume if in each normal equation the second member were equal to the coefficient of $x$. The system of equations so formed would evidently be satisfied by $x = 1, y = 0, z = 0, \ldots t = 0$; hence $\Sigma aa = 1$. In like manner, comparing equation (5) with the same equation, we see that $\Sigma ba$ is the value which $x$ would assume if the second member of each normal equation were equal to the coefficient of $y$. This value would be zero; thus $\Sigma ba = 0$.

† If the value of the weight of $x$ alone is required, it may be found as the reciprocal of what the value of $x$ becomes when $A = 1$, $B = 0$, $C = 0, \ldots$, that is to say, when the second member of the first normal equation is replaced by unity, and that of each of the others by zero.

The weights of $x$, $y$ and $z$ are therefore

$$p_x = \frac{19899}{809} = 24.60,$$

$$p_y = \frac{737}{54} = 13.65,$$

$$p_z = \frac{6633}{123} = 53.93.$$

**133.** When the value of $x$ is obtained by the method of substitution, the process may be so arranged that its weight shall be found at the same time. Let the other unknown quantities be eliminated successively by means of the other normal equations, the value of $x$ being obtained from the first normal equation or normal equation for $x$. Then, if this equation has not been reduced by multiplication or division, the coefficient of $A$ in the second member will still be unity, and the equation will be of the form

$$Rx = T + A,$$

where $T$ depends upon the quantities $B$, $C$, ... Now it is shown in the preceding article that the weight of $x$ is the reciprocal of the coefficient of $A$ in the value of $x$; hence in the present form of the equation the weight is the coefficient of $x$.*

As an illustration, let us find the values of $x$ and its weight in the example given above, the normal equation being

$$\begin{aligned} 27x + 6y &= 88, \\ 6x + 15y + z &= 70, \\ y + 54z &= 107. \end{aligned}$$

The last equation gives

$$z = -\frac{1}{54}y + \frac{107}{54},$$

* The effect of the substitution is always to diminish the coefficient of $x$; for, as mentioned in the foot-note to Art. 122, if the true values of $y, z, \ldots t$ were known, the weight of $x$ would be $\Sigma a^2$, which is the original coefficient of $x$, and obviously the weight on this hypothesis would exceed $p_x$, which is the weight when $y, z, \ldots t$ are also subject to error.

and if this is substituted in the second, we obtain

$$y = -\frac{324}{809}x + \frac{3673}{809}.$$

Finally, by the substitution of this value of $y$ in the first normal equation, we obtain, before any reduction is made,

$$\frac{19899}{809}x = \frac{49154}{809};$$

whence

$$p_x = \frac{19899}{809}, \quad \text{and} \quad x = \frac{49154}{19899},$$

as before found.

*The Determination of the Measure of Precision.*

**134.** The most probable value of $h$ in the case of observations of equal weight is that which gives the greatest possible value to $P$, Art. 120, that is, to the function

$$h^m e^{-h^2(u_1^2+u_2^2+\ldots+u_m^2)},$$

in which the errors are denoted by $u_1, u_2, \ldots u_m$, so that we may retain $v_1, v_2, \ldots v_m$ to denote the residuals which correspond to the values of the unknown quantities derived from the normal equations. By differentiation we derive, as in Art. 69, for the determination of $h$,

$$\Sigma u^2 = \frac{m}{2h^2}. \quad \ldots\ldots\ldots \quad (1)$$

The value of $\Sigma u^2$ cannot, of course, be obtained, but it is known to exceed $\Sigma v^2$, which is its minimum value, and the best value we can adopt is found by adding to $\Sigma v^2$ the mean value of the excess, $\Sigma u^2 - \Sigma v^2$.

**135.** Let the true values of the unknown quantities be $x + \delta x, y + \delta y, \ldots t + \delta t$, while $x, y, \ldots t$ denote the values derived from the normal equations. We have then the residual equations

$$\left.\begin{array}{l} a_1x + b_1y + \ldots + l_1t - n_1 = v_1 \\ a_2x + b_2y + \ldots + l_2t - n_2 = v_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_mx + b_my + \ldots + l_mt - n_m = v_m \end{array}\right\}, \quad . \quad . \quad . \quad (1)$$

and, for the true errors, the expressions,

$$\left.\begin{array}{l} a_1(x + \delta x) + b_1(y + \delta y) + \ldots + l_1(t + \delta t) - n_1 = u_1 \\ a_2(x + \delta x) + b_2(y + \delta y) + \ldots + l_2(t + \delta t) - n_2 = u_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_m(x + \delta x) + b_m(y + \delta y) + \ldots + l_m(t + \delta t) - n_m = u_m \end{array}\right\}. \quad (2)$$

Multiplying equations (1) by $v_1, v_2, \ldots v_m$ respectively, and adding, the coefficient of $x$ in the result is

$$a_1v_1 + a_2v_2 + \ldots + a_mv_m,$$

which vanishes by the first normal equation (1), Art. 121. In like manner, the coefficient of $y$ vanishes by the second normal equation, and so on. Hence

$$\Sigma v^2 = -\Sigma nv. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Treating equations (2) in the same way, we have

$$\Sigma uv = -\Sigma nv;$$

hence

$$\Sigma v^2 = \Sigma uv. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Again, multiplying equations (1) by $u_1, u_2, \ldots u_m$, and adding,

$$\Sigma uv = \Sigma au \,.\, x + \Sigma bu \,.\, y + \ldots + \Sigma lu \,.\, t - \Sigma nu;$$

and treating equations (2) in the same way,

$$\Sigma u^2 = \Sigma au\,(x + \delta x) + \Sigma bu\,(y + \delta y) + \ldots + \Sigma lu\,(t + \delta t) - \Sigma nu.$$

Subtracting the preceding equation, we have, by equation (4),

$$\Sigma u^2 - \Sigma v^2 = \Sigma au \,.\, \delta x + \Sigma bu \,.\, \delta y + \ldots + \Sigma lu \,.\, \delta t, \quad . \quad (5)$$

an expression for the correction whose mean value we are seeking.

**136.** Expressions for $\delta x, \delta y, \ldots \delta t$ are readily obtained as follows. Treating equations (2) exactly as the residual equations (1) are treated to form the normal equations, we find

$$\left.\begin{aligned} &\Sigma a^2.(x+\delta x)+\Sigma ab.(y+\delta y)+\ldots \\ &\qquad\qquad +\Sigma al.(t+\delta t)=\Sigma an+\Sigma au \\ &\Sigma ab.(x+\delta x)+\Sigma b^2.(y+\delta y)+\ldots \\ &\qquad\qquad +\Sigma bl.(t+\delta t)=\Sigma bn+\Sigma bu \\ &\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\Sigma al.(x+\delta x)+\Sigma bl.(y+\delta y)+\ldots \\ &\qquad\qquad +\Sigma l^2.(t+\delta t)=\Sigma ln+\Sigma lu \end{aligned}\right\}.$$

Subtraction of the corresponding normal equation from each of these gives the system,

$$\left.\begin{aligned} &\Sigma a^2.\delta x+\Sigma ab.\delta y+\ldots+\Sigma al.\delta t=\Sigma au \\ &\Sigma ab.\delta x+\Sigma b^2.\delta y+\ldots+\Sigma bl.\delta t=\Sigma bu \\ &\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\Sigma al.\delta x+\Sigma bl.\delta y+\ldots+\Sigma l^2.\delta t=\Sigma lu \end{aligned}\right\},$$

a comparison of which with the normal equations shows that $\delta x, \delta y, \ldots \delta t$ are the same functions of $u_1, u_2, \ldots u_m$ that $x, y, \ldots t$ are of $n_1, n_2, \ldots n_m$. Hence we have

$$\delta x = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_m u_m,$$

where $\alpha_1, \alpha_2, \ldots \alpha_n$ have the same meaning as in Art. 130.

**137.** Consider now the first term, $\Sigma au.\delta x$, of the value of $\Sigma u^2 - \Sigma v^2$, equation (5), Art. 135. Multiplying the value of $\delta x$ just found by

$$\Sigma au = a_1 u_1 + a_2 u_2 + \ldots + a_m u_m,$$

the product consists of terms containing squares and products of the errors. We are concerned only with the mean values of these terms, in accordance with the law of facility, which is for each error $\frac{h}{\sqrt{\pi}} e^{-h^2 u^2}$. Since the mean value of each error is zero, it is obvious that the mean value of each product vanishes;

so that the mean value of $\Sigma au \,.\, \delta x$ is the mean value of

$$a_1 a_1 u_1^2 + a_2 a_2 u_2^2 + \ldots + a_m a_m u_m^2 .$$

Now by Art. 50 the mean value of each of the squares $u_1^2, u_2^2, \ldots u_m^2$ is $\frac{1}{2h^2}$; hence the mean value of $\Sigma au \,.\, \delta x$ is $\frac{\Sigma aa}{2h^2}$, or, by equation (4), Art. 131, $\frac{1}{2h^2}$.

In the same manner it can be shown that the mean value of each term in the second member of equation (5), Art. 135, is $\frac{1}{2h^2}$; hence that of $\Sigma u^2 - \Sigma v^2$ is $\frac{\mu}{2h^2}$, and the best value we can adopt for $\Sigma u^2$ is

$$\Sigma u^2 = \Sigma v^2 + \frac{\mu}{2h^2} .$$

Substituting this in equation (1), Art. 134, we have

$$\Sigma v^2 = \frac{m - \mu}{2h^2}, \qquad \text{whence} \qquad h = \sqrt{\frac{m - \mu}{2\Sigma v^2}} .$$

*The Probable Errors of the Observations and Unknown Quantities.*

**138.** The resulting values of the mean and probable error of a single observation are

$$\varepsilon = \frac{1}{h\sqrt{2}} = \sqrt{\frac{\Sigma v^2}{m - \mu}}, \quad \ldots \quad (1)$$

$$r = \frac{\rho}{h} = \rho\sqrt{2}\sqrt{\frac{\Sigma v^2}{m - \mu}} = 0.6745\sqrt{\frac{\Sigma v^2}{m - \mu}}; \quad . \quad (2)$$

and the probable errors of the unknown quantities are

$$r_x = \frac{r}{\sqrt{p_x}}, \quad r_y = \frac{r}{\sqrt{p_y}}, \quad \ldots \quad r_t = \frac{r}{\sqrt{p_t}}. \quad . \quad (3)$$

When the observation equations have not equal weights we

may replace $\Sigma v^2$, which represents the sum of the squares of the residuals in the reduced equations, by $\Sigma pv^2$, in which the residuals are derived from the original observation equations. The formulæ (1) and (2) will then give the mean and probable errors of an observation whose weight is unity.

It will be noticed that when $\mu = 1$ the formulæ reduce to those given in Art. 72 for the case of one unknown quantity.

139. Instead of calculating the values of $v_1, v_2, \ldots v_m$ directly from the residual equations, and squaring and adding the results, we may employ the formula for $\Sigma v^2$ deduced below.

By equation (3), Art. 135,

$$\Sigma v^2 = -\Sigma nv.$$

Now multiplying equations (1) of that article by $n_1, n_2, \ldots n_m$ respectively, and adding the results, we have

$$\Sigma nv = \Sigma an.x + \Sigma bn.y + \ldots + \Sigma ln.t - \Sigma n^2.$$

Therefore

$$\Sigma v^2 = \Sigma n^2 - \Sigma an.x - \Sigma bn.y - \ldots - \Sigma ln.t. \quad . \quad (1)$$

The quantity $\Sigma n^2$ which occurs in this formula may be calculated at the same time with the coefficients in the normal equations. It enters with them into the check equations of Art. 127.

We may also express $\Sigma v^2$ exclusively in terms of these quantities, for if we write

$$D_n = \begin{vmatrix} \Sigma a^2 & \Sigma ab & \ldots & \Sigma al & \Sigma an \\ \Sigma ab & \Sigma b^2 & \ldots & \Sigma bl & \Sigma bn \\ . & . & & . & . \\ . & . & & . & . \\ . & . & & . & . \\ \Sigma al & \Sigma bl & \ldots & \Sigma l^2 & \Sigma ln \\ \Sigma an & \Sigma bn & \ldots & \Sigma ln & \Sigma n^2 \end{vmatrix},$$

and consider the development of $D_n$ in terms of the elements of its last row, we see that

$$D_n = -\Sigma an.D_x - \Sigma bn.D_y - \ldots - \Sigma ln.D_t + \Sigma n^2.D,$$

where $D$, $D_x$, . . $D_t$ have the same meanings as in Art. 128; hence

$$\Sigma v^2 = \frac{D_n}{D} \cdot \quad . \quad . \quad . \quad . \quad (2)$$

**140.** For example, in the case of the four observation equations of Art. 115,

$$\left.\begin{aligned} x - y + 2z &= 3 \\ 3x + 2y - 5z &= 5 \\ 4x + y + 4z &= 21 \\ -x + 3y + 3z &= 14 \end{aligned}\right\},$$

for which the normal equations are solved in Art. 123, the value of $\Sigma n^2$ is 671; and formula (1) gives

$$\Sigma v^2 = 671 - 88 \times \frac{49154}{19899} - 70 \times \frac{70659}{19899} - 107 \times \frac{38121}{19899} = \frac{1600}{19899},$$

in which 1600 is the value of $D_n$. Substituting this value of $\Sigma v^2$ in the formulæ of Art. 138, we find

$$\varepsilon = 0.2836, \qquad r = 0.1913$$

for the mean and probable errors of an observation; and using the weights found in Art. 132, we find for those of the unknown quantities

$$\begin{aligned} \varepsilon_x &= 0.057, & \varepsilon_y &= 0.077, & \varepsilon_z &= 0.039, \\ r_x &= 0.038, & r_y &= 0.052, & r_z &= 0.026. \end{aligned}$$

In this example we have found the exact value of $\Sigma v^2$; if approximate computations are employed, the formula used has the disadvantage that a very small quantity is to be found by means of large positive and negative terms, which considerably increases the number of significant figures to which the work must be carried. Thus, because $\Sigma n^2 = 671$ in the above example, the work would have to be carried out with seven-place logarithms to obtain $\Sigma v^2$ to four decimal places. The direct

computation of the $v^2$'s from the observation equations would present the same difficulty in a less degree.

**141.** Of course, no great confidence can be placed in the absolute values of the probable errors obtained from so small a number of observation equations as in the example given above. There being but one more observation than barely sufficient to determine values of the unknown quantities, the case is comparable to that in which $n = 2$ when the observations are direct.

By increasing the number of observations we not only obtain a more trustworthy determination of the probable error of a single observation, but, what is more important, we increase the weight, and hence the precision, of the unknown quantities. The measure in which this takes place depends greatly upon the character of the equations with respect to independence. As already mentioned in Art. 113, if there were only $\mu$ equations it would be necessary that they should be independent; in other words, the determinant of their coefficients must not vanish, otherwise the values of the unknown quantities will be indeterminate. When this state of things is approached the values are ill-determined, and this is indicated by the small value of the determinant in question. The same thing is true of the normal equations. Accordingly, the weights are small when the determinant $D$ is small; thus the value of $D$ is in a general way a measure of the efficiency of the system of observation equations in determining the unknown quantities.

**142.** If we write the coefficients in the $m$ observation equations in a rectangular form, thus,

$$\begin{array}{ccccccc} a_1 & a_2 & \dots & a_\mu & \dots & a_m \\ b_1 & b_2 & \dots & b_\mu & \dots & b_m \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ l_1 & l_2 & \dots & l_\mu & \dots & l_m \end{array}$$

the determinant $D$ is, by a theorem in determinants, the sum of the squares of all the determinants which can be formed by

selecting $\mu$ columns of the rectangular array. The first of these determinants is that of the coefficients of the first $\mu$ equations, which, as we have seen, vanishes when they are not independent, and the others are the like determinants for all the other combinations of $\mu$ equations which can be formed from the $m$ observation equations. It follows that $D$ cannot be negative, and cannot vanish unless there is no set of $\mu$ independent equations among the observation equations.

**143.** By a similar consideration of the values of $D_x$, $D_y$, . . . $D_t$, Art. 128, it has been shown * that, for each unknown quantity, the value given by the normal equations is the weighted mean of all the values which could be derived from $\mu$ selected equations, the weights being the squares of the corresponding determinants.†

### *Empirical or Interpolation Formulæ.*

**144.** A set of observation equations usually arises in the following manner: One of two varying quantities is a function of another, *of known form*, the constants which occur having, however, unknown values. Simultaneous values of the varying quantities are observed. The values of the second quantity (the independent variable in the functional expression) are regarded as accurate, and from them are computed in each case the values of the coefficients when the other variable is treated as a linear function of the unknown quantities. This other variable is then the observed quantity $M$ of our observation equations, and the errors are the differences between the observed values and those which accurately correspond to the assumed values of the independent variable.

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* J. W. L. Glaisher, *Monthly Notices of the Royal Ast. Soc.*, vol. xl, 1880, p. 607 et seq.

† When there is but a single unknown quantity, say $x$, its coefficients $a_1, a_2, \ldots$ take the place of these determinants, and the weight of the result is accordingly $\Sigma a^2$. Compare Art. 122. In general, as between two unknown quantities, the weight of that which has the greater coefficients will be the greater.

Taking the two variable quantities as coordinates, the observations may be represented by points, and the problem before us is that of determining a curve of known variety in such a manner as to pass as nearly as possible through these points.

**145.** But it may happen that, while we know that a functional relation between the variable quantities exists, we have no theoretic knowledge of the form of the function. In such cases, our only resource is to assume the form of the function, being guided therein by an inspection of the points representing the observations. An equation so assumed is sometimes called *an empirical formula.* The constants involved in it are determined exactly as in the case of formulæ having a theoretical basis. The final result can only be judged of by the residuals. If these are numerous enough, their failure to follow the law of accidental errors may indicate the inadequacy of the assumed form.

When the formula as determined is used to compute the probable values of the observed quantity corresponding to other values of the independent variable, it is called *an interpolation formula.* The results can never be satisfactory except for values within the range of the values corresponding to the observations upon which the formula is based.

### *Conditioned Observations.*

**146.** We have hitherto supposed the unknown quantities to be independent of one another, so that any set of simultaneous values is possible, and before the observations all sets are regarded as equally probable. It frequently happens, however, that the unknown quantities are required to satisfy rigorously certain *equations of condition*, in addition to the observation equations which must be approximately satisfied. The $\mu$ unknown quantities may thus be subject to $\nu$ equations of condition, where $\nu < \mu$, while the whole number of equations $m + \nu$ exceeds $\mu$. The case may be reduced to that already discussed by the elimination of $\mu'$ unknown quantities from the observation equations by means of the equations of condition, leaving us with $m$

observation equations containing $\mu - \nu$ independent unknown quantities.

We shall consider only the case (which is of frequent occurrence) in which $m = \mu$, and the observation equations express direct determinations of the $\mu$ unknown quantities.

**147.** Let $M_1, M_2, \ldots M_\mu$ be the observed values of $X, Y, \ldots T$, with weights $p_1, p_2, \ldots p_\mu$, and put

$$X = M_1 + x, \quad Y = M_2 + y, \quad \ldots \quad T = M_\mu + t,$$

so that $x, y, \ldots t$ are the required corrections to the observed values. The equations of condition may be reduced as in Art. 117 to the linear forms

$$\left.\begin{array}{l} a_1 x + a_2 y + \ldots + a_\mu t = E_1 \\ b_1 x + b_2 y + \ldots + b_\mu t = E_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ f_1 x + f_2 y + \ldots + f_\mu t = E_\nu \end{array}\right\} \quad . \ . \ . \ . \quad (1)$$

The values of $x, y, \ldots t$ must satisfy these equations, which are, however, insufficient in number to determine them, and, by the principle of Least Squares, those values are most probable which, while satisfying equations (1), make

$$p_1 x^2 + p_2 y^2 + \ldots + p_\mu t^2 = \text{a minimum.}$$

In other words, the values must be such that

$$p_1 x dx + p_2 y dy + \ldots + p_\mu t dt = 0, \quad . \ . \ . \quad (2)$$

for all possible simultaneous values of $dx, dy, \ldots dt$, that is, for all values which satisfy the equations,

$$\left.\begin{array}{l} a_1 dx + a_2 dy + \ldots + a_\mu dt = 0 \\ b_1 dx + b_2 dy + \ldots + b_\mu dt = 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ f_1 dx + f_2 dy + \ldots + f_\mu dt = 0 \end{array}\right\}, \quad . \ . \ . \quad (3)$$

derived by differentiating equations (1). Hence, denoting the

first member of equation (2) by $P$ and those of equations (3) by $S_1, S_2, \ldots S_\nu$, the conditions are fulfilled by values which satisfy equations (1) and make

$$P - k_1 S_1 - k_2 S_2 - \ldots - k_\nu S_\nu = 0, \quad . \quad . \quad . \quad (4)$$

where $k_1, k_2, \ldots k_\nu$ are any constants.

This last equation will be satisfied if we can equate to zero the coefficient of each of the differentials, thus putting

$$\left.\begin{array}{l} p_1 x = k_1 a_1 + k_2 b_1 + \ldots + k_\nu f_1 \\ p_2 y = k_1 a_2 + k_2 b_2 + \ldots + k_\nu f_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ p_\mu t = k_1 a_\mu + k_2 b_\mu + \ldots + k_\nu f_\mu \end{array}\right\}, \quad . \quad . \quad . \quad (5)$$

and this it is possible to do because we have $\mu$ unknown quantities and $\nu$ auxiliary quantities $k_1, k_2, \ldots k_\nu$ which can be determined so as to satisfy the $\nu + \mu$ equations comprised in the groups (1) and (5).

148. Substituting the values of $x, y, \ldots t$ from equations (5) in equations (1), we have a set of linear equations to determine the $k$'s which are called the *correlatives* of the equations of condition. These equations may be written in the form

$$\left.\begin{array}{l} k_1 \Sigma \dfrac{a^2}{p} + k_2 \Sigma \dfrac{ab}{p} + \ldots + k_\nu \Sigma \dfrac{af}{p} = E_1 \\ k_1 \Sigma \dfrac{ab}{p} + k_2 \Sigma \dfrac{b^2}{p} + \ldots + k_\nu \Sigma \dfrac{bf}{p} = E_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ k_1 \Sigma \dfrac{af}{p} + k_2 \Sigma \dfrac{bf}{p} + \ldots + k_\nu \Sigma \dfrac{f^2}{p} = E_\nu \end{array}\right\}, \quad . \quad (6)$$

in which the summation refers to the coefficients of the several unknown quantities; thus, for example, $\Sigma \dfrac{a^2}{p}$ is the sum of the squares of all the coefficients in the first equation of condition each divided by the weight of the corresponding unknown quantity. The correlatives being found from these equations,

the values of the corrections $x, y, \ldots t$ are given at once by equations (5).

**149.** When there is but one equation of condition

$$a_1 x + a_2 y + \ldots + a_\mu t = E,$$

the second members of equations (5) reduce to their first terms, and the equations require that the corrections of the several unknown quantities shall be proportional to their coefficients in the equation of condition divided by their weights. Equations (6) then reduce to the single equation

$$k\Sigma \frac{a^2}{p} = E,$$

and the corrections are

$$x = \frac{\frac{a_1}{p_1}}{\Sigma \frac{a^2}{p}} E, \qquad y = \frac{\frac{a_2}{p_2}}{\Sigma \frac{a^2}{p}} E, \qquad \ldots$$

In the very common case in which the numerical value of each coefficient in the single equation of condition is unity (for example, when the successive angles at a point, or all the angles of a polygon, are measured, or when the sum of two measured angles is independently measured), we have the simple rule that *the corrections are inversely proportional to the weights.*

*Examples.*

1. Denoting the heights above mean sea level of five points by $X$, $Y$, $Z$, $U$, $V$, observations of difference of level gave, in feet:

| | | |
|---|---|---|
| $X = 573.08$ | $Z - Y = 167.33$ | $U - V = 425.00$ |
| $Y - X = 2.60$ | $U - Z = 3.80$ | $V = 319.91$ |
| $Y = 575.27$ | $U - Y = 170.28$ | $V = 319.75$ |

Putting $X = 573 + x$, $Y = 575 + y$, $Z = 742 + z$, $U = 745 + u$,

$V = 320 + v$, find the values and probable errors of the corrections $x, y, z, u, v$, supposing the observations to have equal weight.

$$x = -0.19 \pm 0.23, \quad y = 0.14 \pm 0.21, \quad z = 0.05 \pm 0.30,$$
$$u = 0.43 \pm 0.25, \quad v = 0.03 \pm 0.19.$$

2. Given the observation equations:

$$x = 4.5, \qquad y = 1.6, \qquad x - y = 2.7,$$

with weights 10, 5 and 3 respectively, determine the values of $x$ and $y$. $x = 4.468 \pm 0.049$, $y = 1.663 \pm 0.063$.

3. Measurements of the ordinates of a straight line corresponding to the abscissas 4, 6, 8 and 9, gave the values 5, 8, 10 and 12. What is the most probable equation of the line in the form $y = mx + b$? $y = 1.339x - 0.029$.

4. Given the observation equations of equal weight:

$$x = 10, \qquad y - x = 7, \qquad y = 18,$$
$$y - z = 9, \qquad x - z = 2,$$

determine the most probable values of the unknown quantities, and the probable errors of an observation and of each unknown quantity. $x = 10\frac{3}{8}$, $y = 17\frac{5}{8}$, $z = 8\frac{1}{2}$,
$r = r_z = 0.29$, $r_x = r_y = 0.23$.

5. In order to determine the length $x$ at 0° C. of a meter bar, and its expansion $y$ for each degree of temperature, it was measured at temperatures 20°, 40°, 50°, 60°, the corresponding observed lengths being 1000.22, 1000.65, 1000.90 and 1001.05 mm. respectively. Find the probable values of $x$ and $y$ with their probable errors. $x = 999^{mm}.804 \pm 0.033$,
$y = 0^{mm}.0212 \pm 0.0007$.

6. The length of the pendulum which beats seconds is known to vary with the latitude in accordance with Clairant's equation,

$$l = l' + \left(\frac{5}{2}q - \mu\right) l' \sin^2 L,$$

where $l'$ is the length at the equator, $q$ the ratio $\frac{1}{289}$ of the cen-

trifugal force at the equator to the weight, and $\mu$ the compression of the meridian regarded as unknown. Putting

$$l' = 991^{mm} + x, \qquad \left(\frac{5}{2}q - \mu\right) l' = y,$$

observations in different latitudes gave in millimeters:

$x + 0.969y = 5.13$, $x + 0.095y = 0.56$, $x + 0.327y = 1.70$,
$x + 0.749y = 3.97$, $x = 0.19$, $x + 0.685y = 3.62$,
$x + 0.426y = 2.24$, $x + 0.152y = 0.77$, $x + 0.793y = 4.23$.

Find the length at the equator with its probable error.

$l' = 991^{mm}.069 \pm .026$.

7. Find the value of $\mu$ in the preceding example and its probable error. $\mu = \frac{1}{294} \pm 0.00046$.

8. The measured height in feet of $A$ above $O$, $B$ above $A$ and $B$ above $O$ are 12.3, 14.1 and 27.0 respectively. Find the most probable value and the probable error of each of these differences of level. $12.5 \pm 0.17$; $14.3 \pm 0.17$; $26.8 \pm 0.17$.

9. A round of angles at a station in the U. S. Coast Survey was observed with weights as follows:

65° 11′ 52″.500 with weight 3, 87° 2′ 24″.703 with weight 3,
66 24 15 .553 " " 3, 141 21 21 .757 " " 1;

find the adjusted values whose sum must be 360°.

65° 11′ 53″.4145, 87° 2′ 25″.6175,
66 24 16 .4675, 141 21 24 .5005.

10. Four observations on the angle $X$ of a triangle gave a mean of 36° 25′ 47″, two observations on $Y$ gave a mean of 90° 36′ 28″ and three on $Z$ gave 52° 57′ 57″. Find the adjusted values of the angles and the probable error of a single observation. $r = 7''.7$; $X = 36°\ 25'\ 44''.23$,
$Y = 90\ 36\ 22\ .46$,
$Z = 52\ 57\ 53\ .31$.

11. A round of four angles was observed as follows:

38° 52′ 14″.28 weight 2, 44° 35′ 56″.54 weight 3,
145 23 16 .35 " 4, 131 10 21 .47 " 3,

find the adjusted values.

38° 51′ 35″.94, 44° 35′ 30″.98,
145 22 57 .18, 131 9 55 .91.

12. Measurements of the angles between surrounding stations were made with weights as follows:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Between stations | 1 | and | 2, | 55° | 57′ | 58″.68, | weight | 3, |
| " | " | 2 | " | 3, | 48 | 49 | 13 .64, | " | 19, |
| " | " | 1 | " | 3, | 104 | 47 | 12 .66, | " | 17, |
| " | " | 3 | " | 4, | 54 | 38 | 15 .53, | " | 13, |
| " | " | 2 | " | 4, | 103 | 27 | 28 .99, | " | 6. |

Find the corrections of the angles in the order given.

0″.285, 0″.005, − 0″.050, − 0″.058, 0″.127.

# IX.

## Gauss's Method of Substitution.

### *The Reduced Normal Equations.*

**150.** In solving the normal equations, it becomes essential, except in the simplest cases, to reduce the labor as much as possible by adopting a systematic process in the elimination. We shall here give the method of substitution as developed by Gauss, which has the advantage of preserving, in each of the sets of simultaneous equations which arise in the elimination, the symmetry which exists in the coefficients of the normal equations, thereby materially diminishing the number of coefficients to be calculated.

The $m$ observation equations, involving the $\mu$ unknown quantities $x, y, z, \ldots t$, being, as in Art. 124,

$$\left.\begin{aligned} a_1x + b_1y + \ldots + l_1t &= n_1 \\ a_2x + b_2y + \ldots + l_2t &= n_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_mx + b_my + \ldots + l_mt &= n_m \end{aligned}\right\}, \quad \ldots \quad (1)$$

let the normal equations be written in the form

$$\left.\begin{aligned} [aa]x + [ab]y + [ac]z + \ldots + [al]t &= [an] \\ [ab]x + [bb]y + [bc]z + \ldots + [bl]t &= [bn] \\ [ac]x + [bc]y + [cc]z + \ldots + [cl]t &= [cn] \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ [al]x + [bl]y + [cl]z + \ldots + [ll]t &= [ln] \end{aligned}\right\} \ldots (2)$$

As mentioned at the end of Art. 126, we may suppose the observation equations (1) to have been reduced to the weight unity, so that $[aa]$, $[ab]$, $\ldots$ $[ln]$ stand for $\Sigma a^2$, $\Sigma ab$, $\ldots$ $\Sigma ln$.

**151.** The value of $x$ in terms of the other unknown quantities derived from the first of equations (2), or normal equation for $x$, is

$$x = -\frac{[ab]}{[aa]}y - \frac{[ac]}{[aa]}z - \ldots + \frac{[an]}{[aa]}.$$

Substituting this in the $\mu - 1$ other equations, they become

$$\left.\begin{aligned}
\left([bb] - [ab]\frac{[ab]}{[aa]}\right)y + \left([bc] - [ab]\frac{[ac]}{[aa]}\right)z + \ldots &= [bn] - [ab]\frac{[an]}{[aa]} \\
\left([bc] - [ac]\frac{[ab]}{[aa]}\right)y + \left([cc] - [ac]\frac{[ac]}{[aa]}\right)z + \ldots &= [cn] - [ac]\frac{[an]}{[aa]} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\left([bl] - [al]\frac{[ab]}{[aa]}\right)y + \ldots + \left([ll] - [al]\frac{[al]}{[aa]}\right)t &= [ln] - [al]\frac{[an]}{[aa]}
\end{aligned}\right\},$$

in which it will be noticed that the coefficients of the unknown quantities have the same symmetry as in the normal equations (2). These equations for the $\mu - 1$ unknown quantities $y, z, \ldots t$ are called *the reduced normal equations*, and are written in the form

$$\left.\begin{aligned}
[bb, 1]y + [bc, 1]z + \ldots + [bl, 1]t &= [bn, 1] \\
[bc, 1]y + [cc, 1]z + \ldots + [cl, 1]t &= [cn, 1] \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
[bl, 1]y + [cl, 1]z + \ldots + [ll, 1]t &= [ln, 1]
\end{aligned}\right\}, \quad . \quad (3)$$

in which

$$\left.\begin{aligned}
[bb, 1] &= [bb] - \frac{[ab][ab]}{[aa]} \\
[bc, 1] &= [bc] - \frac{[ab][ac]}{[aa]} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
[ln, 1] &= [ln] - \frac{[al][an]}{[aa]}
\end{aligned}\right\} \cdot \cdot \cdot \cdot \cdot \quad (4)$$

Equations (4) show that the rule for the formation of the coefficients and the second members of the reduced normal equations is the same throughout; namely, from the corresponding coefficient in the normal equations we are to subtract the result of multiplying together the two expressions in whose symbols one of the letters in the given symbol is associated with $a$, and dividing the product by $[aa]$.

### *The Elimination Equations.*

**152.** Eliminating $y$ by means of the first of the reduced normal equations (3) from each of the others, just as $x$ was eliminated from the normal equations, and employing a similar notation, we have the $\mu - 2$ equations

$$\left.\begin{aligned} [cc, 2]z + \ldots + [cl, 2]t &= [cn, 2] \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ [cl, 2]z + \ldots + [ll, 2]t &= [ln, 2] \end{aligned}\right\}, \quad \ldots \quad (5)$$

which may be called the *second reduced normal equations.* The coefficients in these equations are derived from those in equations (3) exactly as the latter were found from those in equations (2). Thus

$$\left.\begin{aligned} [cc, 2] &= [cc, 1] - \frac{[bc, 1][bc, 1]}{[bb, 1]} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ [ln, 2] &= [ln, 1] - \frac{[bl, 1][bn, 1]}{[bb, 1]} \end{aligned}\right\}. \quad \ldots \quad (6$$

In like manner the third reduced normal equations are formed from these last, the coefficients being distinguished by the postfixed numeral 3, corresponding to the number of

variables which have been eliminated. We finally arrive at the single equation

$$[ll, \mu - 1]t = [ln, \mu - 1], \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which determines the unknown quantity standing last in the order of elimination.

**153.** The quantity which immediately precedes $t$ is next derived from the first of the preceding set of equations (that is, from the equation by means of which it was eliminated) by the substitution of the numerical value found for $t$; and so on, until finally $x$ is found from the first of the original normal equations. The equations from which the unknown quantities are actually determined are therefore the following:

$$\left.\begin{array}{r} [aa]x + [ab]y + [ac]z + \ldots + [al]t = [an] \\ [bb, 1]y + [bc, 1]z + \ldots + [bl, 1]t = [bn, 1] \\ [cc, 2]z + \ldots + [cl, 2]t = [cn, 2] \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ [ll, \mu - 1]t = [ln, \mu - 1] \end{array}\right\} . \quad (8)$$

These are called the *final* or *elimination equations.*

*The Reduced Observation Equations.*

**154.** Let us suppose that there exists a relation between the variables which must be exactly satisfied, while the $m$ observation equations are to be satisfied approximately. Let this relation be

$$\alpha x + \beta y + \ldots + \lambda t = \nu. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Eliminating $x$ from the observation equations (1), Art. 150 by the substitution of

$$x = -\frac{\beta}{\alpha}y - \frac{\gamma}{\alpha}z - \ldots - \frac{\lambda}{\alpha}t + \frac{\nu}{\alpha},$$

derived from this equation, we have

$$\left.\begin{aligned}
\left(b_1-a_1\frac{\beta}{\alpha}\right)y+\left(c_1-a_1\frac{\gamma}{\alpha}\right)z+\ldots+\left(l_1-a_1\frac{\lambda}{\alpha}\right)t&=n_1-a_1\frac{\nu}{\alpha}\\
\left(b_2-a_2\frac{\beta}{\alpha}\right)y+\left(c_2-a_2\frac{\gamma}{\alpha}\right)z+\ldots+\left(l_2-a_2\frac{\lambda}{\alpha}\right)t&=n_2-a_2\frac{\nu}{\alpha}\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot&\\
\left(b_m-a_m\frac{\beta}{\alpha}\right)y+\left(c_m-a_m\frac{\gamma}{\alpha}\right)z+\ldots+\left(l_m-a_m\frac{\lambda}{\alpha}\right)t&=n_m-a_m\frac{\nu}{\alpha}
\end{aligned}\right\},$$

which may be called *the reduced observation equations*, and written in the form

$$\left.\begin{aligned}
b_1'y+c_1'z+\ldots+l_1't&=n_1'\\
b_2'y+c_2'z+\ldots+l_2't&=n_2'\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot&\\
b_m'y+c_m'z+\ldots+l_m't&=n_m'
\end{aligned}\right\},\quad\ldots\quad(2)$$

a comparison of which with the equations written above sufficiently indicates the values of $b_1', c_1', \ldots a_m', \ldots n_m'$.

The $\mu - 1$ normal equations derived from these are

$$\left.\begin{aligned}
[b'b']y+[b'c']z+\ldots+[b'l']t&=[b'n']\\
[b'c']y+[c'c']z+\ldots+[c'l']t&=[c'n']\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot&\\
[b'l']y+[c'l']z+\ldots+[l'l']t&=[l'n']
\end{aligned}\right\},\quad\ldots\quad(3)$$

in which

$$\left.\begin{aligned}
[b'b']&=\Sigma\left(b-a\frac{\beta}{\alpha}\right)^2&&=[bb]-2[ab]\frac{\beta}{\alpha}+[aa]\frac{\beta^2}{\alpha^2}\\
[b'c']&=\Sigma\left(b-a\frac{\beta}{\alpha}\right)\left(c-a\frac{\gamma}{\alpha}\right)&&=[bc]-[ab]\frac{\gamma}{\alpha}-[ac]\frac{\beta}{\alpha}+[aa]\frac{\beta\gamma}{\alpha^2}\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot&&&\\
[l'n']&=\Sigma\left(l-a\frac{\lambda}{\alpha}\right)\left(n-a\frac{\nu}{\alpha}\right)&&=[ln]-[al]\frac{\nu}{\alpha}-[an]\frac{\lambda}{\alpha}+[aa]\frac{\lambda\nu}{\alpha^2}
\end{aligned}\right\}.(4)$$

155. Let us now suppose that the equation of condition (1) which is to be exactly satisfied is identical with the first of the normal equations (2) of Art 150, so that

$$\alpha = [aa], \quad \beta = [ab], \quad \ldots \quad \nu = [an];$$

then equations (4) become

$$\left.\begin{aligned} [b'b'] &= [bb] - \frac{[ab]^2}{[aa]} \\ [b'c'] &= [bc] - \frac{[ab][ac]}{[aa]} \\ &\cdots\cdots \\ [l'n'] &= [ln] - \frac{[al][an]}{[aa]} \end{aligned}\right\} \quad \ldots\ldots \quad (5)$$

Comparison of these with equations (4), Art. 151, shows that the normal equations (3) of the preceding article now become identical with the first reduced normal equations of Art. 151. Hence *the first reduced normal equations are the same as the normal equations corresponding to the reduced observation equations which would result if $x$ were eliminated from the observation equations by means of the normal equation for $x$.*

It is evident that, in like manner, the second reduced normal equations are the same as the $\mu - 2$ normal equation which would result from the reduced observation equations, if they were further reduced by the elimination of $y$ by means of the reduced normal equation for $y$; or, what is the same thing, the normal equations which would result if $x$ and $y$ were eliminated from the original observation equations by means of the normal equations for $x$ and $y$. Similar remarks apply to the other sets of reduced normal equations.

156. An important consequence of what has just been proved is that, among the coefficients in the reduced normal equations, or *auxiliary quantities*, those of quadratic form,

$$[bb, 1], \quad [cc, 1], \quad \ldots \quad [cc, 2], \quad \ldots \quad [ll, \mu - 1],$$

being, like the corresponding quantities in the normal equations, sums of squares, are all positive. It is further to be noticed that each of these quantities decreases as its postfix increases, for the subtractive quantities in the formation of the successive values are themselves positive. For example,

$$[ll, 1] = [ll] - \frac{[al]^2}{[aa]}, \quad [ll, 2] = [ll, 1] - \frac{[bl, 1]^2}{[bb, 1]}.$$

*Weights of the Two Quantities First Determined.*

**157.** The unknown quantity $t$ has been determined in equation (7), Art. 152, after the manner described in Art. 133; that is to say, from its own normal equation—no reduction by multiplication or division having taken place in the course of the elimination. Hence, as proved in that article, its weight is the coefficient of the unknown quantity; that is to say, the weight of an observation being unity, that of $t$ is

$$p_t = [ll, \mu - 1],$$

which, as shown in the preceding article, is necessarily a positive quantity.*

The weight of any one of the unknown quantities might be determined, in like manner, by making it the last in the order of elimination.

**158.** Let $s$ be the unknown quantity preceding $t$, so that

$$[ll, \mu - 1] = [ll, \mu - 2] - \frac{[kl, \mu - 2]^2}{[kk, \mu - 2]},$$

* As shown in Art. 156, the substitutions diminish the successive coefficients of $t$. Compare the foot-note to Art. 133, p. 104. In fact $[ll]$ is the weight that $t$ would have if the true values of all the other quantities were known; $[ll, 1]$ is the weight which it would have if all the others except $x$ were known—that is, if $x$ and $t$ were the only quantities subject to error; and so on.

or

$$[ll, \mu - 1][kk, \mu - 2] = [ll, \mu - 2][kk, \mu - 2] - [kl, \mu - 2]^2.$$

If now the order of $s$ and $t$ be reversed, no other change of order being made, the auxiliaries with the postfix $\mu - 2$ will be unaltered, and we shall have

$$[kk, \mu - 1][ll, \mu - 2] = [kk, \mu - 2][ll, \mu - 2] - [kl, \mu - 2]^2,$$

hence

$$[kk, \mu - 1][ll, \mu - 2] = [ll, \mu - 1][kk, \mu - 2].$$

But $[kk, \mu - 1]$ is the weight of $s$, therefore we have

$$p_s = [kk, \mu - 1] = \frac{[kk, \mu - 2]}{[ll, \mu - 2]}[ll, \mu - 1].$$

The weights of the other unknown quantities cannot be thus readily expressed in terms of the auxiliaries occurring in the calculation of $t$. A general method of obtaining all the weights will be given in Arts. 174–176.

### *The Reduced Expression for $\Sigma v^2$.*

**159.** We have found in Art. 139 for $\Sigma v^2$ or $[vv]$ the expression

$$[vv] = -[an]x - [bn]y - \ldots - [ln]t + [nn],$$

which is similar in form to the expressions equated to zero in the normal equations. If in this we substitute the value of $x$, as in Art. 151, it becomes

$$[vv] = -[bn, 1]y - [cn, 1]z - \ldots - [ln, 1]t + [nn, 1],$$

in which

$$[bn, 1] = [bn] - \frac{[ab][an]}{[aa]},$$

. . . . .

$$[nn, 1] = [nn] - \frac{[an][an]}{[aa]},$$

after the analogy of the auxiliary quantities defined in equations (4), Art. 151. In like manner, by the elimination of $y$, $[vv]$ is reduced to the form

$$[vv] = -[cn, 2]z - \ldots - [ln, 2]t + [nn, 2],$$

and finally, by the substitution of the value of $t$, to

$$[vv] = [nn, \mu],$$

the postfix $\mu$ indicating that all the unknown quantities have been eliminated.

Substituting in the expressions for the mean and probable error of an observation, Art. 138, we have

$$\epsilon = \sqrt{\frac{[nn, \mu]}{m - \mu}}, \qquad r = 0.6745\sqrt{\frac{[nn, \mu]}{m - \mu}}.$$

*The General Expression for the Sum of the Squares of the Errors.*

**160.** The following articles contain an investigation* of the sum of the squares of the errors considered as a function of the unknown quantities, showing directly that the minimum

* Gauss, "Theoria Motus Corporum Cœlestium," Art. 182; *Werke*, vol. vii. p. 238.

value of this quantity corresponds to the values derived from the normal equations, and is equal to $[nn, \mu]$, and also deriving from the general expression the law of facility of error in $t$, and thence its weight.

Let

$$W = [vv] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be the sum of the squares of the errors in the observation equations, that is to say, of the linear expressions of the form (Art. 119),

$$ax + by + \ldots + lt - n = v.$$

The absolute term in $W$ is obviously $[nn]$. Put

$$\tfrac{1}{2}\frac{dW}{dx} = X, \quad \tfrac{1}{2}\frac{dW}{dy} = Y, \quad \ldots \quad \tfrac{1}{2}\frac{dW}{dt} = T. \quad . \quad (2)$$

Then

$$X = \Sigma v\frac{dv}{dx} = [av] = [aa]x + [ab]y + \ldots + [al]t - [an]. \quad (3)$$

The equations $X = 0$, $Y = 0, \ldots T = 0$ are the normal equations. Now, since

$$\tfrac{1}{2}\frac{d(X^2)}{dx} = X\frac{dX}{dx} = [aa]X, \quad \text{or,} \quad \tfrac{1}{2}\frac{d}{dx}\frac{X^2}{[aa]} = X,$$

$$\tfrac{1}{2}\frac{d}{dx}\left(W - \frac{X^2}{[aa]}\right) = 0;$$

hence, if we put

$$W_1 = W - \frac{X^2}{[aa]}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$W_1$ is a function independent of $x$. Now, in equation (4), $W_1$ has for all values of the variables which make $X = 0$

the same value as $W$; hence $W_1$ is what $W$ becomes when $x$ is eliminated from it by means of the first normal equation, $X = 0$.

**161.** It follows from what has just been proved, that

$$W_1 = [v'v'] ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

that is to say, $W_1$ is the sum of the squares of expressions of the form

$$b'y + c'z + \ldots + l't - n' = v',$$

corresponding to the reduced observation equations, Arts. 154, 155. The absolute term in $W_1$ is therefore $[n'n']$ or $[nn, 1]$.

If, now, we put

$$Y_1 = \frac{1}{2}\frac{dW_1}{dy}, \quad \ldots \quad T_1 = \frac{1}{2}\frac{dW_1}{dt}, \quad . \quad . \quad . \quad (6)$$

$$Y_1 = \Sigma v'\frac{dv'}{dy} = [b'v'] = [b'b']y + [b'c']z + \ldots + [b'l']t - [b'n']. \quad (7)$$

and $Y_1 = 0, \ldots T_1 = 0$, are the reduced normal equations.

The relation between the expressions $Y_1, \ldots T_1$ and $X, Y, \ldots T$ is derived from equation (4); thus, differentiating with respect to $Y$,

$$Y_1 = Y - \frac{X}{[aa]}\frac{dX}{dy} = Y - \frac{[ab]}{[aa]}X, \quad . \quad . \quad . \quad (8)$$

which gives another proof of the identity of the coefficients $[b'b'], \ldots [b'n']$ with $[bb, 1], \ldots [bn, 1]$, established in Art. 155. We now prove, exactly as in the preceding article, that

$$W_2 = W_1 - \frac{Y_1^2}{[bb, 1]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

is a function independent of $y$ as well as of $x$, and is identical

with $[v''v'']$, the sum of the squares of expressions of the form

$$c''z + \ldots + l''t - n'' = v''$$

corresponding to the second reduced observation equations, from which $x$ and $y$ have been eliminated by means of the equations $X = 0$, $Y_1 = 0$. The absolute term in $W_2$ is obviously $[n''n'']$ or $[nn, 2]$.

**162.** Proceeding in this way, we finally arrive at an expression $W_\mu$ which is independent of all the variables, and consists simply of the absolute term $[nn, \mu]$. We have thus reduced $W$ to the form

$$W = \frac{X^2}{[aa]} + \frac{Y_1^2}{[bb, 1]} + \frac{Z_2^2}{[cc, 2]} + \ldots + \frac{T_{\mu-1}^2}{[ll, \mu - 1]} + [nn, \mu].^* \quad (10)$$

The denominators $[aa]$, $[bb, 1], \ldots [ll, \mu - 1]$, being sums of squares, are all positive; hence the minimum value of $W$ is the value $[nn, \mu]$ corresponding to the values of $x, y, \ldots t$ which satisfy the equations $X = 0$, $Y_1 = 0, \ldots T_{\mu-1} = 0$.

**163.** Since $W$ is the sum of the squares of the errors, the probability that the actual observations should occur is proportional to $e^{-h^2W}$ as in Art. 62. Therefore, by the principle explained in Art. 30, the observations having been made, the probabilities of different systems of values of the unknown quantities are proportional to the corresponding values of this function. Hence, $C$ being a constant to be determined, the elementary probability, Art. 21, of a given system of values of $x, y, \ldots t$ is

$$Ce^{-h^2W} dx\, dy \ldots dt, \quad \ldots\ldots \quad (11)$$

* This result is also derived by Gauss in a purely algebraic manner in the "Disquisitio de Elementis Ellipticis Paladis;" *Werke*, vol. vi. p. 22. See also Encke, *Berliner Astronomisches Jahrbuch* for 1853, pp. 273–277.

where $h$ is the measure of precision of an observation, and $C$ is such that the integral of the expression for all possible values of the variables is unity.

The probability of a given system of values of $y, z, \ldots t$, while $x$ may have any value, is found by summing this expression for all values of $x$. It is then

$$C\,dy \ldots dt \int_{-\infty}^{\infty} e^{-h^2 W} dx = C dy \ldots dt\, e^{-h^2 W_1} \int_{-\infty}^{\infty} e^{-h^2 \frac{X^2}{[aa]}} dx,$$

since $W_1$ in equation (4) is independent of $x$. Since $\frac{dX}{dx} = [aa]$, the value of the definite integral in this expression is, by equation (7), Art. 39,

$$\int_{-\infty}^{\infty} e^{-h^2 \frac{X^2}{[aa]}} dx = \frac{1}{[aa]} \int_{-\infty}^{\infty} e^{-\frac{h^2}{[aa]} X^2} dX = \frac{\sqrt{\pi}}{h\sqrt{[aa]}}.$$

Thus the probability of a given system of values of $y$, $z, \ldots t$ is

$$\frac{C\sqrt{\pi}}{h\sqrt{[aa]}} dy dz \ldots dt\, e^{-h^2 W_1}. \quad . \quad . \quad . \quad . \quad (12)$$

**164.** In like manner, the probability of a given system of values of $z \ldots t$, $x$ and $y$ being indeterminate, is

$$\frac{C\sqrt{\pi}}{h\sqrt{[aa]}} dz \ldots dt \int_{-\infty}^{\infty} e^{-h^2 W_1} dy,$$

which, by equations (9) and (7), reduces to

$$\frac{C\sqrt{(\pi^2)}}{h^2\sqrt{\{[aa][bb, 1]\}}} dz \ldots dt\, e^{-h^2 W_2}. \quad . \quad . \quad (13)$$

Proceeding in this way, we have, finally, for the probability of a given value of $t$,

$$\frac{C\sqrt{(\pi^{\mu-1})}dt}{h^{\mu-1}\sqrt{\{[aa][bb,\,1]\ldots[kk,\,\mu-2]\}}}e^{-h^2W_{\mu-1}}. \quad (14)$$

Again, integrating this for all values of $t$, we have

$$C\frac{\sqrt{(\pi^{\mu})}e^{-h^2[nn,\,\mu]}}{h^{\mu}\sqrt{\{[aa][bb,\,1]\ldots[ll,\,\mu-1]\}}} = 1. \quad (15)$$

Substituting the value of $C$ thus determined, we obtain for the probability of $t$,

$$\frac{h\sqrt{[ll,\,\mu-1]}}{\sqrt{\pi}}e^{-h^2(W_{\mu-1}-[nn,\,\mu])}dt. \quad (16)$$

But

$$W_{\mu-1} = \frac{T^2_{\mu-1}}{[ll,\,\mu-1]} + [nn,\,\mu]$$

and

$$T_{\mu-1} = [ll,\,\mu-1]t - [ln,\,\mu-1];$$

therefore, putting

$$\tau = \frac{T_{\mu-1}}{[ll,\,\mu-1]} = t - \frac{[ln,\,\mu-1]}{[ll,\,\mu-1]},$$

and omitting $dt$, the expression (16) gives for the law of facility of error in $t$,

$$\frac{h\sqrt{[ll,\,\mu-1]}}{\sqrt{\pi}}e^{-h^2[ll,\,\mu-1]\tau^2}. \quad (17)$$

This is of the same form as the law of facility for an observation, except that the measure of precision is

$$h\sqrt{[ll, \mu - 1]}.$$

Thus the most probable value of $t$ is that which makes $\tau = 0$, namely,

$$t = \frac{[ln, \mu - 1]}{[ll, \mu - 1]},$$

and the weight of this determination, when that of an observed quantity is unity, is

$$p_t = [ll, \mu - 1].$$

*The Auxiliaries Expressed in Determinant Form.*

**165.** If, in the determinant of the coefficients of the normal equations, denoted by $D$ in Art. 128, we subtract from the second row the product of the first row multiplied by $\frac{[ab]}{[aa]}$, it becomes

$$0, \quad [bb, 1], \quad [bc, 1], \quad \ldots \quad [bl, 1].$$

Treating the other rows in like manner, the determinant $D$ is reduced to a form in which the first row is unchanged, and the rest are replaced by a column of 0's and the determinant of the first reduced normal equations. Denoting this last determinant by $D'$, we have $D = [aa]D'$.

By a similar reduction of $D'$, $D$ is further reduced to a form in which the first two rows are as in that described above, and the rest are replaced by two columns of 0's and the determinant, $D''$, of the second reduced normal equations. Finally, $D$ is thus reduced to the determinant of the elimination equations (8), Art. 153.

The successive forms of $D$ give the equations

$$D=[aa]D'=[aa][bb,1]D''=\ldots=[aa][bb,1][cc,2]\ldots[ll,\mu-1].$$

**166.** If, in the form of $D$ involving $D^{(r)}$, we take the first $r$ rows, and then any other row (which will therefore be a row belonging to $D^{(r)}$), the same reasoning shows that any determinant formed by selecting $r+1$ columns of this rectangular block is equal to the minor occupying the same position in $D$.

We can now express any auxiliary, say $[\alpha\beta, r]$, as the quotient of two minors, of the $(r+1)$th and $r$th degree respectively, in $D$. This auxiliary occurs in the form of $D$ just mentioned. Taking the first $r$ rows and columns together with the row and column in which the given auxiliary occurs, we have a determinant whose value is

$$[aa][bb,1]\ldots[\gamma\gamma, r-1][\alpha\beta, r],$$

because all the elements below the principal diagonal vanish. But this determinant is equal to that similarly situated in $D$, and the coefficient of $[\alpha\beta, r]$ is equal to the determinant formed from the first $r$ rows and columns of $D$. For example, for $[de, 2]$ we have

$$\begin{vmatrix} [aa] & [ab] & [ae] \\ 0 & [bb,1] & [be,1] \\ 0 & 0 & [de,2] \end{vmatrix} = \begin{vmatrix} [aa] & [ab] & [ae] \\ [ab] & [bb] & [be] \\ [ad] & [bd] & [de] \end{vmatrix},$$

and

$$\begin{vmatrix} [aa] & [ab] \\ 0 & [bb,1] \end{vmatrix} = \begin{vmatrix} [aa] & [ab] \\ [ab] & [bb] \end{vmatrix};$$

therefore

$$[de,2]\begin{vmatrix} [aa] & [ab] \\ [ab] & [bb] \end{vmatrix} = \begin{vmatrix} [aa] & [ab] & [ae] \\ [ab] & [bb] & [be] \\ [ad] & [bd] & [de] \end{vmatrix}.$$

**167.** The same principle holds if we include the auxiliaries involving the letter $n$, and in particular the determinant $D_n$ of Art. 139 is

$$D_n = [aa][bb, 1] \ldots [ll, \mu - 1][nn, \mu] = D[nn, \mu];$$

therefore

$$[nn, \mu] = \frac{D_n}{D},$$

which is the same value that was found for $[vv]$ on p. 110.

*Form of the Calculation of the Auxiliaries.*

**168.** In calculating the coefficients which occur in the elimination equations and the value of $[vv]$, it is important to arrange the work in tabular form, and to apply frequent checks to the computation to secure accuracy. In the annexed table,* which is constructed for four unknown quantities, the first compartment contains the coefficients and second members of the normal equations together with the value of $[nn]$, which are derived from the observation equations, as explained in Art. 127. The coefficients are entered opposite and below the letters in their symbols, those below the diagonal line, whose values are the same as those symmetrically situated above, being omitted. Beneath those in the first line are written their logarithms, which are used in computing the subtractive quantities placed beneath each of the other coefficients.

* The tabular arrangement is taken from W. Jordan's "Handbuch der Vermessungskunde." See also Oppolzer's "Lehrbuch zur Bahnbestimmung der Kometen und Planeten," vol. ii. p. 340 *et seq.*, where the table, with a somewhat different arrangement, is given for six unknown quantities, and an example is fully worked out.

| | *a* | *b* | *c* | *d* | *n* | *s* |
|---|---|---|---|---|---|---|
| *a* | $[aa]$<br>log $[aa]$ | $[ab]$<br>log $[ab]$ | $[ac]$<br>log $[ac]$ | $[ad]$<br>log $[ad]$ | $[an]$<br>log $[an]$ | $[as]$<br>log $[as]$ |
| *b* | log $A_b$ | $[bb]$<br>$A_b[ab]$ | $[bc]$<br>$A_b[ac]$ | $[bd]$<br>$A_b[ad]$ | $[bn]$<br>$A_b[an]$ | $[bs]$<br>$A_b[as]$ |
| *c* | log $A_c$ | | $[cc]$<br>$A_c[ac]$ | $[cd]$<br>$A_c[ad]$ | $[cn]$<br>$A_c[an]$ | $[cs]$<br>$A_c[as]$ |
| *d* | log $A_d$ | | | $[dd]$<br>$A_d[ad]$ | $[dn]$<br>$A_d[an]$ | $[ds]$<br>$A_d[as]$ |
| *n* | log $A_n$ | | | | $[nn]$<br>$A_n[an]$ | $[ns]$<br>$A_n[as]$ |
| *b* | | $[bb, 1]$<br>log $[bb, 1]$ | $[bc, 1]$<br>log $[bc, 1]$ | $[bd, 1]$<br>log $[bd, 1]$ | $[bn, 1]$<br>log $[bn, 1]$ | $[bs, 1]$<br>log $[bs, 1]$ |
| *c* | log $B_c$ | | $[cc, 1]$<br>$B_c[bc, 1]$ | $[cd, 1]$<br>$B_c[bd, 1]$ | $[cn, 1]$<br>$B_c[bn, 1]$ | $[cs, 1]$<br>$B_c[bs, 1]$ |
| *d* | log $B_d$ | | | $[dd, 1]$<br>$B_d[bd, 1]$ | $[dn, 1]$<br>$B_d[bn, 1]$ | $[ds, 1]$<br>$B_d[bs, 1]$ |
| *n* | log $B_n$ | | | | $[nn, 1]$<br>$B_n[bn, 1]$ | $[ns, 1]$<br>$B_n[bs, 1]$ |
| *c* | | | $[cc, 2]$<br>log $[cc, 2]$ | $[cd, 2]$<br>log $[cd, 2]$ | $[cn, 2]$<br>log $[cn, 2]$ | $[cs, 2]$<br>log $[cs, 2]$ |
| *d* | log $C_d$ | | | $[dd, 2]$<br>$C_d[cd, 2]$ | $[dn, 2]$<br>$C_d[cn, 2]$ | $[ds, 2]$<br>$C$ $[cs, 2]$ |
| *n* | log $C_n$ | | | | $[nn, 2]$<br>$C_n[cn, 2]$ | $[ns, 2]$<br>$C_n[cs, 2]$ |
| *d* | | | | $[dd, 3]$<br>log $[dd, 3]$ | $[dn, 3]$<br>log $[dn, 3]$ | $[ds, 3]$<br>log $[ds, 3]$ |
| *n* | log $D_n$ = log $t$ | | | | $[nn, 3]$<br>$D_n[dn, 3]$ | $[ns, 3]$<br>$D_n[ds, 3]$ |
| *n* | | $t =$ | $[vv] =$ | | $[nn, 4]$ | $[ns, 4]$ |

In expressing the subtractive quantities we have adopted for abridgment the notation

$$A_b = \frac{[ab]}{[aa]}, \quad A_c = \frac{[ac]}{[aa]}, \quad A_d = \frac{[ad]}{[aa]}, \quad A_n = \frac{[an]}{[aa]}.$$

The logarithms of these quantities are placed at the side, and, adding them successively to the logarithms above, the antilogarithms of the sums are entered in their places. After this is done, the results of subtraction are the auxiliaries with postfix 1, which are to be placed in corresponding positions in the compartment below.

In like manner the third compartment is formed from the second, and in expressing the subtractive quantities we have put

$$B_c = \frac{[bc, 1]}{[bb, 1]}, \quad B_d = \frac{[bd, 1]}{[bb, 1]}, \quad B_n = \frac{[bn, 1]}{[bb, 1]}.$$

So also we have put

$$C_d = \frac{[cd, 2]}{[cc, 2]}, \qquad C_n = \frac{[cn, 2]}{[cc, 2]};$$

and finally,

$$D_n = \frac{[dn, 3]}{[dd, 3]},$$

which is also the value of $t$. Thus the first four compartments correspond to the several sets of normal equations, and their first lines to the four elimination equations. Finally, in the fifth compartment we have computed $[nn, 4]$, which is the value of $[vv]$.

*Check Equations.*

**169.** The column headed $s$ is added for the sake of the check equations

$$\left.\begin{array}{l} [aa] + [ab] + [ac] + [ad] + [an] + [as] = 0 \\ [ab] + [bb] + [bc] + [bd] + [bn] + [bs] = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ [an] + [bn] + [cn] + [dn] + [nn] + [ns] = 0 \end{array}\right\}, \quad . \quad (1)$$

the quantities $[as], \ldots [ns]$ being formed as in Art. 127, except that we have changed the sign of $s$, so that for each observation equation

$$a + b + c + d + n + s = 0.$$

The checks are applied before the logarithms and subtractive quantities are entered. They require that the algebraic sum of the quantities in each line together with those standing above the first term should vanish.

Similar checks can be applied in each of the lower compartments. For example, if from the second of equations (1) we subtract the product of the first equation multiplied by $A_b$, we have, since $A_b[aa] = [ab]$,

$$0 + [bb, 1] + [bc, 1] + [bd, 1] + [bn, 1] + [bs, 1] = 0,$$

where $[bs, 1]$ has been formed in precisely the same way as the other auxiliaries, namely,

$$[bs, 1] = [bs] - \frac{[ab][as]}{[aa]}.$$

In the same manner we obtain the other equations of the group

$$\left.\begin{array}{l} [bb, 1] + [bc, 1] + [bd, 1] + [bn, 1] + [bs, 1] = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ [bn, 1] + [cn, 1] + [dn, 1] + [nn, 1] + [ns, 1] = 0 \end{array}\right\}. \quad (2)$$

So also we have similar checks involving the auxiliaries which have the postfix 2, and those which have the postfix 3, and finally

$$[nn, 4] + [ns, 4] = 0.$$

| | *a* | *b* | *c* | *d* | *n* | *s* | |
|---|---|---|---|---|---|---|---|
| *a* | 3.1217 | .5756 | − .1565 | − .0067 | 1.5710 | −5.1050 | 1 |
| | 0.49439 | 9.76012 | $9_n$19451 | $7_n$82607 | 0.19618 | $0_n$70800 | |
| *b* | | 2.9375 | .1103 | − .0015 | − .9275 | −2.6943 | 1 |
| $A_b$ | 9.26573 | .1061 | − .0289 | − .0012 | .2897 | − .9413 | |
| *c* | | | 4.1273 | .2051 | − .0652 | −4.2211 | −1 |
| $A_c$ | $8_n$70012 | | .0078 | .0003 | − .0788 | .2559 | |
| *d* | | | | 4.1328 | − .0178 | −4.3118 | 1 |
| $A_d$ | $7_n$33168 | | | .0000 | − .0034 | .0110 | |
| *n* | | | | | 1.3409 | −1.9016 | −2 |
| $A_n$ | 9.70179 | | | | .7906 | −2.5692 | |
| *b* | | 2.8314 | .1392 | − .0003 | −1.2172 | −1.7530 | 1 |
| | | 0.45200 | 9.14364 | $6_n$47712 | $0_n$08536 | $0_n$24378 | |
| *c* | | | 4.1195 | .2048 | .0136 | −4.4770 | 1 |
| $B_c$ | 8.69164 | | .0068 | .0000 | − .0598 | − .0862 | |
| *d* | | | | 4.1328 | − .0144 | −4.3228 | 1 |
| $B_d$ | $6_n$02512 | | | .0000 | .0001 | .0002 | |
| *n* | | | | | .5503 | .6676 | −1 |
| $B_n$ | $9_n$63336 | | | | .5233 | .7536 | |
| *c* | | | 4.1127 | .2048 | .0734 | −4.3908 | 1 |
| | | | 0.61413 | 9.31133 | 8.86570 | $0_n$64254 | |
| *d* | | | | 4.1328 | − .0145 | −4.3230 | 1 |
| $C_d$ | 8.69720 | | | .0102 | .0037 | − .2186 | |
| *n* | | | | | .0270 | − .0860 | −1 |
| $C_n$ | 8.25157 | | | | .0013 | − .0784 | |
| *d* | | | | 4.1226 | − .0182 | −4.1044 | 0 |
| | | | | 0.61517 | $8_n$26007 | $0_n$61325 | |
| *n* | | | | | .0257 | − .0076 | −1 |
| $D_n$ | $7_n$64490 = log *t* | | | | ·0001 | .0181 | |
| *n* | *t* = − .004415 | | [*vv*] = .02565 | | .0256 | − .0257 | −1 |

*Numerical Example.*

**170.** As an illustration, let us take the following normal equations:

$$\left.\begin{aligned} 3.1217x + .5756y - .1565z - .0067t &= 1.5710 \\ .5756x + 2.9375y + .1103z - .0015t &= -.9275 \\ -.1565x + .1103y + 4.1273z + .2051t &= -.0652 \\ -.0067x - .0015y + .2051z + 4.1328t &= -.0178 \end{aligned}\right\}$$

together with

$$[nn] = 1.3409,$$

which were derived from sixteen observation equations, while at the same time the values of $[as], \ldots [ns]$ were found as in the first compartment of the table. The numbers in the final column are the sums which should equal zero according to the check equations, the small errors being due to the rejection of decimals beyond the fourth place. The letters at the side and top indicate the symbol for each auxiliary, while the compartment gives the postfix. Since there are two computations for $[vv]$, namely $[nn, 4]$ and $-[ns, 4]$, which agree within the limits of the uncertainty of logarithmic computation, we take for its value a mean between them. Putting $m = 16$ and $\mu = 4$ in the formulæ for $\epsilon$ and $r$, this value gives

$$\epsilon = .04623, \qquad r = .03118,$$

for the mean and probable error of an observation.

*Values of the Unknown Quantities from the Elimination Equations.*

**171.** Dividing the elimination equations, (8), Art. 153, by $[aa]$, $[bb, 1]$, $[cc, 2]$, $[dd, 3]$, and using the notation introduced in Art. 168, they become

$$\left.\begin{aligned} x + A_b y + A_c z + A_d t &= A_n \\ y + B_c z + B_d t &= B_n \\ z + C_d t &= C_n \\ t &= D_n \end{aligned}\right\} \quad . \quad . \quad . \quad (1,$$

The following table gives the form in which the computation is conveniently arranged, and its application to the example for which the elimination equations are found in Art. 170.

| $D_n$ | $C_n$<br>$-C_d t$ | $B_n$<br>$-B_d t$<br>$-B_c z$ | $A_n$<br>$-A_d t$<br>$-A_c z$<br>$-A_b y$ |
|---|---|---|---|
| $t$<br>log $t$ | $z$<br>log $z$ | $y$<br>log $y$ | $x$ |
| −.004415 | .017847<br>.000220 | −.42989<br>.00000<br>−.00089 | .50325<br>−.00001<br>.00091<br>.07943 |
| −.004415<br>$7_n64490$ | .018067<br>8.25689 | −.43078<br>$9_n63426$ | .58358 |

The weight of $t$ is, by Art. 157, $[dd, 3]$, and that of $z$ is, by Art. 158, $\frac{[cc, 2]}{[dd, 2]}[dd, 3]$; employing the values computed in Art. 170, we have

$$\log p_t = 0.61517, \qquad \log p_z = 0.61305,$$
$$p_t = 4.1226, \qquad p_z = 4.1025;$$

and dividing the values of $\epsilon$ and $t$ found above by the square roots of the weights, we have for $t$,

$$\epsilon_t = .02277, \qquad r_t = .01536;$$

and for $z$,

$$\epsilon_z = .02282, \qquad r_z = .01539.$$

*Values of the Unknown Quantities Found Independently.*

172. In order to obtain the general expressions for the weights, it is necessary first to express the values of the unknown quantities independently of each other. For this pur-

pose we multiply equations (1) of the preceding article by 1, $\alpha_1$, $\alpha_2$, $\alpha_3$, respectively, and add the results, assuming the $\alpha$'s to be so determined that the coefficients of $y$, $z$, and $t$ vanish. We shall thus have

$$x = A_n + B_n\alpha_1 + C_n\alpha_2 + D_n\alpha_3, \quad . \quad . \quad . \quad (2)$$

and, for the determination of the $\alpha$'s,

$$\left.\begin{aligned} A_b + \alpha_1 &= 0 \\ A_c + B_c\alpha_1 + \alpha_2 &= 0 \\ A_d + B_d\alpha_1 + C_d\alpha_2 + \alpha_3 &= 0 \end{aligned}\right\} . \quad . \quad . \quad (3)$$

In like manner, to find $y$ we multiply the second, third and fourth of equations (1) by 1, $\beta_2$, $\beta_3$, respectively, and add. The result is

$$y = B_n + C_n\beta_2 + D_n\beta_3, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where the $\beta$'s are determined by

$$\left.\begin{aligned} B_c + \beta_2 &= 0 \\ B_d + C_d\beta_2 + \beta_3 &= 0 \end{aligned}\right\} . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Again, multiplying the last two of equations (1) by 1, $\gamma_3$, and adding

$$z = C_n + D_n\gamma_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $\gamma_3$ is determined by

$$C_d + \gamma_3 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

**173.** The form for the computation of $\alpha_1$, $\alpha_2$, $\alpha_3$, $\beta_2$, $\beta_3$, $\gamma_3$, according to equations (3), (5), and (7), and the numerical work for the example of Art. 170, is as follows:

| $-A_b$ | $-A_c$<br>$-B_c\alpha_1$ | $-A_d$<br>$-B_d\alpha_1$<br>$-C_d\alpha_2$ |
|---|---|---|
| $\alpha_1$<br>$\log \alpha_1$ | $\alpha_2$<br>$\log \alpha_2$ | $\alpha_3$<br>$\log \alpha_3$ |
| | .050133<br>.009065 | .002146<br>−.000020<br>−.002948 |
| | .059198 | −.000822 |
| $9_n26573$ | 8.77231 | $6_n91487$ |

| $-B_c$ | $-B_d$<br>$-C_d\beta_2$ |
|---|---|
| $\beta_2$<br>$\log \beta_2$ | $\beta_3$<br>$\log \beta_3$ |
| | .000106<br>.002448 |
| | .002554 |
| $8_n69164$ | 7.40722 |

$-C_d = \gamma_3$
$\log \gamma_3 = 8_n69720$

The values of $\alpha_1$, $\beta_2$ and $\gamma_3$ are not found, as their logarithms only are needed.

We may now recompute the values of the unknown quantities by means of equations (2), (4), (6) by way of verifying the values of $\alpha_1, \ldots \gamma_3$ as well as those of $x, \ldots t$. The form of computation will be as below:

| $A_n$<br>$B_n\alpha_1$<br>$C_n\alpha_2$<br>$D_n\alpha_3$ | $B_n$<br>$C_n\beta_2$<br>$D_n\beta_3$ | $C_n$<br>$D_n\gamma_3$ | $D_n$ |
|---|---|---|---|
| $x$ | $y$ | $z$ | $t$ |
| .50325<br>.07927<br>.00106<br>.00000 | −.42989<br>−.00088<br>−.00001 | .017847<br>.000220 | −.004415 |
| .58358 | −.43078 | .018067 | −.004415 |

The numerical values agree exactly with those found in Art. 171.

*The Weights of the Unknown Quantities.*

174. The principle by which we obtain expressions for the weights is that proved in Art. 132, namely: When the value of any one of the unknown quantities is expressed in terms of the second members of the normal equations, its weight is the reciprocal of the coefficient of the second member of its own normal equation; or what is the same thing: *The reciprocal of the weight is what the value of the unknown quantity becomes when the second member of its own normal equation is replaced by unity and that of each of the others by zero.*

Restoring the values of the quantities $A_n$, $B_n$, $C_n$, $D_n$, the values of $x, y, z, t$, Art. 172, are

$$\left.\begin{aligned} x &= \frac{[an]}{[aa]} + \frac{[bn,\ 1]}{[bb,\ 1]}\alpha_1 + \frac{[cn,\ 2]}{[cc,\ 2]}\alpha_2 + \frac{[dn,\ 3]}{[dd,\ 3]}\alpha_3 \\ y &= \frac{[bn,\ 1]}{[bb,\ 1]} + \frac{[cn,\ 2]}{[cc,\ 2]}\beta_2 + \frac{[dn,\ 3]}{[dd,\ 3]}\beta_3 \\ z &= \frac{[cn,\ 2]}{[cc,\ 2]} + \frac{[dn,\ 3]}{[dd,\ 3]}\gamma_3 \\ t &= \frac{[dn,\ 3]}{[dd,\ 3]} \end{aligned}\right\}. \qquad (1)$$

Equations (3), (5), and (7), Art. 172, show that the values of $\alpha_1, \ldots \gamma_3$ are independent of the values of $[an]$, $[bn]$, $[cn]$, and $[dn]$; hence the changes indicated above, in order to convert the second members of equations (1) into the expressions for the reciprocals of the weights, have only to be made in the numerators $[an]$, $[bn,\ 1]$, $[cn,\ 2]$, and $[dn,\ 3]$, where, by the definitions given in Arts. 151 and 152, we have, using the notation of Art. 168,

$$\left.\begin{aligned} [bn, 1] &= [bn] - A_b[an] \\ [cn, 2] &= [cn] - A_c[an] - B_c[bn, 1] \\ [dn, 3] &= [dn] - A_d[an] - B_d[bn, 1] - C_d[cn, 2] \end{aligned}\right\}. \quad (2)$$

175. To find the value of $\frac{1}{p_x}$, we must now put in the value of $x$

$$[an] = 1, \qquad [bn] = 0, \qquad [cn] = 0, \qquad [dn] = 0.$$

Making these substitutions and using equations (3), Art. 172, the value of $[bn, 1]$ becomes

$$[bn, 1] = -A_b = \alpha_1;$$

that of $[cn, 2]$ then becomes

$$[cn, 2] = -A_c - B_c\alpha_1 = \alpha_2;$$

and that of $[dn, 3]$ becomes

$$[dn, 3] = -A_d - B_d\alpha_1 - C_d\alpha_2 = \alpha_3.$$

Hence from the first of equations (1) we infer

$$\frac{1}{p_x} = \frac{1}{[aa]} + \frac{\alpha_1^2}{[bb, 1]} + \frac{\alpha_2^2}{[cc, 2]} + \frac{\alpha_3^2}{[dd, 3]}.$$

176. Again, to obtain the weight of $y$, we put in the second of equations (1)

$$[an] = 0, \qquad [bn] = 1, \qquad [cn] = 0, \qquad [dn] = 0.$$

These substitutions in equations (2) give, with the aid of equations (5), Art. 172,

$$[bn, 1] = 1,$$
$$[cn, 2] = -B_c = \beta_2,$$
$$[dn, 3] = -B_d - C_d\beta_2 = \beta_3;$$

hence we have

$$\frac{1}{p_n} = \frac{1}{[bb, 1]} + \frac{\beta_2^2}{[cc, 2]} + \frac{\beta_3^2}{[dd, 3]}.$$

In like manner we complete the system of equations

$$\left.\begin{aligned}
\frac{1}{p_x} &= \frac{1}{[aa]} + \frac{\alpha_1^2}{[bb, 1]} + \frac{\alpha_2^2}{[cc, 2]} + \frac{\alpha_3^2}{[dd, 3]} \\
\frac{1}{p_y} &= \qquad\quad \frac{1}{[bb, 1]} + \frac{\beta_2^2}{[cc, 2]} + \frac{\beta_3^2}{[dd, 3]} \\
\frac{1}{p_z} &= \qquad\qquad\quad \frac{1}{[cc, 2]} + \frac{\gamma_3^2}{[dd, 3]} \\
\frac{1}{p_t} &= \qquad\qquad\qquad\quad \frac{1}{[dd, 3]}
\end{aligned}\right\}, \quad \cdots \quad (3)$$

which are readily extended to the case of any number of unknown quantities.

**177.** The form of computation and its application to our numerical example are given on page 148, the values of the logarithms entered at the top and side being taken from the computations on pages 140 and 144.

From the logarithms in the last line and $\log \epsilon^2 = 7.32990$, ($\epsilon^2 = \frac{1}{12}[vv]$, p. 140) we find for the mean errors

$$\epsilon_x = .02669, \quad \epsilon_y = .02750, \quad \epsilon_z = .02283, \quad \epsilon_t = .02277;$$

and hence for the probable errors

$$r_x = .01800, \quad r_y = .01855, \quad r_z = .01539, \quad r_t = .01536.$$

| $\log \alpha_1^2$<br>$\log \alpha_2^2$<br>$\log \alpha_3^2$ | $\log \beta_2^2$<br>$\log \beta_3^2$ | $\log \gamma_3^2$ | |
|---|---|---|---|
| $\frac{1}{[aa]}$ | | | $\log \frac{1}{[aa]}$ |
| $\frac{\alpha_1^2}{[bb,\ 1]}$ | $\frac{1}{[bb,\ 1]}$ | | $\log \frac{1}{[bb,\ 1]}$ |
| $\frac{\alpha_2^2}{[cc,\ 2]}$ | $\frac{\beta_2^2}{[cc,\ 2]}$ | $\frac{1}{[cc,\ 2]}$ | $\log \frac{1}{[cc,\ 2]}$ |
| $\frac{\alpha_3^2}{[dd,\ 3]}$ | $\frac{\beta_3^2}{[dd,\ 3]}$ | $\frac{\gamma_3^2}{[dd,\ 3]}$ | $\log \frac{1}{[dd,\ 3]}$ |
| $\frac{1}{p_x}$ | $\frac{1}{p_y}$ | $\frac{1}{p_z}$ | $\frac{1}{p_t}$ |
| $\log \frac{1}{p_x}$ | $\log \frac{1}{p_y}$ | $\log \frac{1}{p_z}$ | $\log \frac{1}{p_t}$ |
| 8.53146 | | | |
| 7.54462 | 7.38328 | | |
| 3.82974 | 4.81444 | 7.39440 | |
| .32034 | | | 9.50561 |
| .01201 | .35318 | | 9.54800 |
| .00085 | .00059 | .24315 | 9.38587 |
| .00000 | .00000 | .00060 | 9.38483 |
| .33320 | .35377 | .24375 | |
| 9.52270 | 9.54872 | 9.38694 | 9.38483 |

*Examples.*

1. Show that the values of $p_z$ when there are four unknown quantities given in Arts. 158 and 176 are identical.

2. Show that the weight of the determination of $[bn]$ is $[bb]$; that of $[bn, 1]$ is $[bb, 1]$, and so on.

3. Show that, if the normal equation for $x$ were known to be exactly true, the values of the unknown quantities and the weights relatively to that of an observation of all except $x$ would be unchanged, and that the weight of an observation would be increased in the ratio $m - \mu + 1 : m - \mu$.

4. Solve the following normal equations which resulted from twelve observation equations:

$$\begin{aligned} 5.1143x - 0.2792y + 3.3460z &= -0.7365, \\ -0.2792x + 14.6142y + 0.1958z &= 2.1609, \\ 3.3460x + 0.1958y + 7.6754z &= -0.8927, \\ [nn] &= 0.5379, \end{aligned}$$

and find the probable errors of the unknown quantities.

$$x = -.0803, \quad y = .1475, \quad z = .0851;$$
$$r_x = .034, \quad r_y = .017, \quad r_z = .028.$$

5. Solve the normal equations

$$\begin{aligned} 5.2485x - 1.7472y - 2.1954z &= -0.5399, \\ -1.7472y + 1.8859y + 0.8041z &= 1.4493, \\ -2.1954y + 0.8041y + 4.0440z &= 1.8681, \\ [nn] &= 2.6322; \end{aligned}$$

and given $m = 10$, find the probable errors.

$$[vv] = 0.5504, \quad x = 0.422, \quad y = 0.945, \quad z = 0.503;$$
$$r = 0.189, \quad r_x = 0.108, \quad r_y = 0.166, \quad r_z = 0.107.$$

6. Show that the observation equations

$$\begin{aligned}
0.707x + 2.052y - 2.372z - 0.221t &= -\ 6.58,\\
0.471x + 1.347y - 1.715z - 0.085t &= -\ 1.63,\\
0.260x + 0.770y - 0.356z + 0.483t &=\ \ 4.40,\\
0.092x + 0.343y + 0.235z + 0.469t &=\ \ 10.21,\\
0.414x + 1.204y - 1.506z - 0.205t &= -\ 3.99,\\
0.040x + 0.150y + 0.104z + 0.206t &=\ \ 4.34,
\end{aligned}$$

give rise to the normal equations

$$\begin{aligned}
0.971x + 2.821y - 3.175z - 0.104t &= -\ 4.815,\\
2.821x + 8.208y - 9.168z - 0.251t &= -\ 12.961,\\
-\ 3.175x - 9.168y + 11.028z + 0.938t &=\ \ 25.697,\\
-\ 0.104x - 0.251y + 0.938z + 0.594t &=\ \ 10.218,
\end{aligned}$$

and to $[nn] = 204.313$. Determine the unknown quantities and the probable errors of an observation.

$x = -\ 86.41$, $y = 25.18$, $z = -\ 3.12$, $t = 17.66$, $r = 1.80$.

7. Account for the small values of the weights, especially of $x$ and $y$, in Ex. 6. Show directly from the value of $[bb, 1]$ that $p_y < .012$ and $p_x < .0015$.

8. Ten observation equations gave the normal equations

$$\begin{aligned}
2.02530x + 0.63809y - 3.99285z &= -\ 30.466,\\
0.63809x + 0.21649y - 1.12089z &= -\ 11.959,\\
-\ 3.99285x - 1.12089y + 10.00000z &= -\ 6.000,
\end{aligned}$$

together with $[nn] = 24928.$; find the values and weights of the unknown quantities and the probable errors.

$$x = -202.8, \quad y = 286.3, \quad z = -49.5;$$
$$p_x = .0314, \quad p_y = .0066, \quad p_z = .9119;$$
$$r = 37.702, \quad r_x = 213, \quad r_y = 463, \quad r_z = 39.$$

9. Given the following observation equations of equal weight:

$$.986x + .056y = .000, \qquad .953x + .182y = 1.060,$$
$$.973x + .103y = .530, \qquad .943x + .219y = -.380,$$
$$.968x + .123y = .680, \qquad .919x + .307y = .200,$$
$$.959x + .157y = .200, \qquad .916x + .317y = -.530,$$
$$.912x + 331y = .000,$$

find the normal equations and the value of $[nn]$ by the method of Art. 127. (Notice that when we put $a + b + n + s = 0$ as in Art. 169 a considerable saving of labor results from the fact that $\Sigma(a+b)^2 = \Sigma(n+s)^2$, etc.)

$$8.0884x + 1.6798y = 1.7160,$$
$$1.6798x + 0.4383y = 0.1725,$$
$$[nn] = 2.3722.$$

10. Solve the normal equations found in Ex. 9.

$$x = 0.642, \quad y = -2.07, \quad r_x = 0.25, \quad r_y = 1.09.$$

11. Thirteen observation equations give the normal equations

$$17.50x - 6.50y - 6.50z = 2.14,$$
$$-6.50x + 17.50y - 6.50z = 13.96,$$
$$-6.50x - 6.50y + 20.50z = -5.40,$$
$$[nn] = 100.34;$$

find the values and probable errors of the unknown quantities.

$$x = 0.67 \pm 0.60, \quad y = 1.17 \pm 0.60, \quad z = 0.32 \pm 0.55.$$

12. Solve the normal equations

$$\begin{aligned} 459x - 308y - 389z + 244t &= 507, \\ -308x + 464y + 408z - 269t &= -695, \\ -389x + 408y + 676z - 331t &= -653, \\ 244x - 269y - 331z + 469t &= 283, \\ [nn] &= 1129. \end{aligned}$$

$x = -0.212$, $y = -1.471$, $z = -0.195$, $t = -0.488$; $[vv] = 10$; $p_x = 207$, $p_y = 186$, $p_z = 250$, $p_t = 281$.

*Constants.*

$$\rho = 0.4769352, \quad \log \rho = 9.6784603;$$
$$\rho\sqrt{2} = 0.6744897, \quad \log \rho\sqrt{2} = 9.8289753;$$
$$\rho\sqrt{\pi} = 0.8453475, \quad \log \rho\sqrt{\pi} = 8.9270353;$$
$$r = \rho\sqrt{2} \,.\, \epsilon = \rho\sqrt{\pi} \,.\, \eta.$$

Note that $\rho\sqrt{2} = \alpha + \beta + \gamma + \delta + \ldots$, where $\alpha = \frac{2}{3}$, $\beta = \frac{1}{100}\alpha$, $\gamma = \frac{1}{6}\beta$, $\delta = \frac{4}{100}\gamma$.

# VALUES OF THE PROBABILITY INTEGRAL,

## OR PROBABILITY OF AN ERROR NUMERICALLY LESS THAN $x$.

### TABLE I.—VALUES OF $P_t$.

$$t = hx;\ P_t = \frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \operatorname{Erf} t.$$

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0000 | 0.0113 | 0.0226 | 0.0338 | 0.0451 | 0.0564 | 0.0676 | 0.0789 | 0.0901 | 0.1013 |
| 0.1 | 1125 | 1236 | 1348 | 1459 | 1569 | 1680 | 1790 | 1900 | 2009 | 2118 |
| 0.2 | 2227 | 2335 | 2443 | 2550 | 2657 | 2763 | 2869 | 2974 | 3079 | 3183 |
| 0.3 | 0.3286 | 0.3389 | 0.3491 | 0.3593 | 0.3694 | 0.3794 | 0.3893 | 0.3992 | 0.4090 | 0.4187 |
| 0.4 | 4284 | 4380 | 4475 | 4569 | 4662 | 4755 | 4847 | 4937 | 5027 | 5117 |
| 0.5 | 5205 | 5292 | 5379 | 5465 | 5549 | 5633 | 5716 | 5798 | 5879 | 5959 |
| 0.6 | 0.6039 | 0.6117 | 0.6194 | 0.6270 | 0.6346 | 0.6420 | 0.6494 | 0.6566 | 0.6638 | 0.6708 |
| 0.7 | 6778 | 6847 | 6914 | 6981 | 7047 | 7112 | 7175 | 7238 | 7300 | 7361 |
| 0.8 | 7421 | 7480 | 7538 | 7595 | 7651 | 7707 | 7761 | 7814 | 7867 | 7918 |
| 0.9 | 0.7969 | 0.8019 | 0.8068 | 0.8116 | 0.8163 | 0.8209 | 0.8254 | 0.8299 | 0.8342 | 0.8385 |
| 1.0 | 8427 | 8468 | 8508 | 8548 | 8586 | 8624 | 8661 | 8698 | 8733 | 8768 |
| 1.1 | 8802 | 8835 | 8868 | 8900 | 8931 | 8961 | 8991 | 9020 | 9048 | 9076 |
| 1.2 | 0.9103 | 0.9130 | 0.9155 | 0.9181 | 0.9205 | 0.9229 | 0.9252 | 0.9275 | 0.9297 | 0.9319 |
| 1.3 | 9340 | 9361 | 9381 | 9400 | 9419 | 9438 | 9456 | 9473 | 9490 | 9507 |
| 1.4 | 9523 | 9539 | 9554 | 9569 | 9583 | 9597 | 9611 | 9624 | 9637 | 9649 |
| 1.5 | 0.9661 | 0.9673 | 0.9684 | 0.9695 | 0.9706 | 0.9716 | 0.9726 | 0.9736 | 0.9745 | 0.9755 |
| 1.6 | 9763 | 9772 | 9780 | 9788 | 9796 | 9804 | 9811 | 9818 | 9825 | 9832 |
| 1.7 | 9838 | 9844 | 9850 | 9856 | 9861 | 9867 | 9872 | 9877 | 9882 | 9886 |
| 1.8 | 0.9891 | 0.9895 | 0.9899 | 0.9903 | 0.9907 | 0.9911 | 0.9915 | 0.9918 | 0.9922 | 0.9925 |
| 1.9 | 9928 | 9931 | 9934 | 9937 | 9939 | 9942 | 9944 | 9947 | 9949 | 9951 |
| 2.0 | 9953 | 9955 | 9957 | 9959 | 9961 | 9963 | 9964 | 9966 | 9967 | 9969 |
| 2.1 | 0.9970 | 0.9972 | 0.9973 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9979 | 0.9980 | 0.9980 |
| 2.2 | 9981 | 9982 | 9983 | 9984 | 9985 | 9985 | 9986 | 9987 | 9987 | 9988 |
| 2.3 | 9989 | 9989 | 9990 | 9990 | 9991 | 9991 | 9992 | 9992 | 9992 | 9993 |
| 2.4 | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 | 0.9995 | 0.9996 |
| 2.5 | 9996 | 9996 | 9996 | 9997 | 9997 | 9997 | 9997 | 9997 | 9997 | 9998 |
| 2.6 | 9998 | 9998 | 9998 | 9998 | 9998 | 9998 | 9998 | 9998 | 9998 | 9999 |
| 2.7 | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| 2.8 | ... | ... | ... | ... | ... | ... | 0.9999 | 1.0000 | ... | ... |

## TABLE II.—VALUES OF $P_z$.

$$z = \frac{x}{r} = \frac{t}{\rho};\ P_z = \frac{2}{\sqrt{\pi}} \operatorname{Erf} \rho z = \frac{2}{\sqrt{\pi}} \int_0^{\rho z} e^{-t^2} dt.$$

| $z$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0000 | 0.0054 | 0.0108 | 0.0161 | 0.0215 | 0.0269 | 0.0323 | 0.0377 | 0.0430 | 0.0484 |
| 0.1 | 0538 | 0591 | 0645 | 0699 | 0752 | 0806 | 0859 | 0913 | 0966 | 1020 |
| 0.2 | 1073 | 1126 | 1180 | 1233 | 1286 | 1339 | 1392 | 1445 | 1498 | 1551 |
| 0.3 | 0.1604 | 0.1656 | 0.1709 | 0.1761 | 0.1814 | 0.1866 | 0.1919 | 0.1971 | 0.2023 | 0.2075 |
| 0.4 | 2127 | 2179 | 2230 | 2282 | 2334 | 2385 | 2436 | 2488 | 2539 | 2590 |
| 0.5 | 2641 | 2691 | 2742 | 2793 | 2843 | 2893 | 2944 | 2994 | 3044 | 3093 |
| 0.6 | 0.3143 | 0.3192 | 0.3242 | 0.3291 | 0.3340 | 0.3389 | 0.3438 | 0.3487 | 0.3535 | 0.3584 |
| 0.7 | 3632 | 3680 | 3728 | 3775 | 3823 | 3871 | 3918 | 3965 | 4012 | 4059 |
| 0.8 | 4105 | 4152 | 4198 | 4244 | 4290 | 4336 | 4381 | 4427 | 4472 | 4517 |
| 0.9 | 0.4562 | 0.4606 | 0.4651 | 0.4695 | 0.4739 | 0.4783 | 0.4827 | 0.4871 | 0.4914 | 0.4957 |
| 1.0 | 5000 | 5043 | 5085 | 5128 | 5170 | 5212 | 5254 | 5295 | 5337 | 5378 |
| 1.1 | 5419 | 5460 | 5500 | 5540 | 5581 | 5621 | 5660 | 5700 | 5739 | 5778 |
| 1.2 | 0.5817 | 0.5856 | 0.5894 | 0.5932 | 0.5971 | 0.6008 | 0.6046 | 0.6083 | 0.6121 | 0.6157 |
| 1.3 | 6194 | 6231 | 6267 | 6303 | 6339 | 6375 | 6410 | 6445 | 6480 | 6515 |
| 1.4 | 6550 | 6584 | 6618 | 6652 | 6686 | 6719 | 6753 | 6786 | 6818 | 6851 |
| 1.5 | 0.6883 | 0.6915 | 0.6947 | 0.6979 | 0.7011 | 0.7042 | 0.7073 | 0.7104 | 0.7134 | 0.7165 |
| 1.6 | 7195 | 7225 | 7255 | 7284 | 7313 | 7343 | 7371 | 7400 | 7428 | 7457 |
| 1.7 | 7485 | 7512 | 7540 | 7567 | 7594 | 7621 | 7648 | 7675 | 7701 | 7727 |
| 1.8 | 0.7753 | 0.7778 | 0.7804 | 0.7829 | 0.7854 | 0.7879 | 0.7904 | 0.7928 | 0.7952 | 0.7976 |
| 1.9 | 8000 | 8023 | 8047 | 8070 | 8093 | 8116 | 8138 | 8161 | 8183 | 8205 |
| 2.0 | 8227 | 8248 | 8270 | 8291 | 8312 | 8332 | 8353 | 8373 | 8394 | 8414 |
| 2.1 | 0.8433 | 0.8453 | 0.8473 | 0.8492 | 0.8511 | 0.8530 | 0.8549 | 0.8567 | 0.8585 | 0.8604 |
| 2.2 | 8622 | 8639 | 8657 | 8674 | 8692 | 8709 | 8726 | 8743 | 8759 | 8776 |
| 2.3 | 8792 | 8808 | 8824 | 8839 | 8855 | 8870 | 8886 | 8901 | 8916 | 8930 |
| 2.4 | 0.8945 | 0.8959 | 0.8974 | 0.8988 | 0.9002 | 0.9016 | 0.9029 | 0.9043 | 0.9056 | 0.9069 |
| 2.5 | 9082 | 9095 | 9108 | 9121 | 9133 | 9146 | 9158 | 9170 | 9182 | 9194 |
| 2.6 | 9205 | 9217 | 9228 | 9239 | 9250 | 9261 | 9272 | 9283 | 9293 | 9304 |
| 2.7 | 0.9314 | 0.9324 | 0.9334 | 0.9344 | 0.9354 | 0.9364 | 0.9373 | 0.9383 | 0.9392 | 0.9401 |
| 2.8 | 9411 | 9419 | 9428 | 9437 | 9446 | 9454 | 9463 | 9471 | 9479 | 9487 |
| 2.9 | 9495 | 9503 | 9511 | 9519 | 9526 | 9534 | 9541 | 9548 | 9556 | 9563 |
| 3.0 | 0.9570 | 0.9577 | 0.9583 | 0.9590 | 0.9597 | 0.9603 | 0.9610 | 0.9616 | 0.9622 | 0.9629 |
| 3.1 | 9635 | 9641 | 9647 | 9652 | 9658 | 9664 | 9669 | 9675 | 9680 | 9686 |
| 3.2 | 9691 | 9696 | 9701 | 9706 | 9711 | 9716 | 9721 | 9726 | 9731 | 9735 |
| 3.3 | 0.9740 | 0.9744 | 0.9749 | 0.9753 | 0.9757 | 0.9762 | 0.9766 | 0.9770 | 0.9774 | 0.9778 |
| 3.4 | 9782 | 9786 | 9789 | 9793 | 9797 | 9800 | 9804 | 9807 | 9811 | 9814 |
| 3. | 9570 | 9635 | 9691 | 9740 | 9782 | 9818 | 9848 | 9874 | 9896 | 9915 |
| 4. | 0.9930 | 0.9943 | 0.9954 | 0.9963 | 0.9970 | 0.9976 | 0.9981 | 0.9985 | 0.9988 | 0.9991 |
| 5. | 9993 | 9994 | 9995 | 9996 | 9997 | 9998 | 9998 | 9999 | 9999 | 9999 |
| 6. | 9999 | 1.0000 | ... | ... | ... | ... | ... | ... | ... | ... |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 1 | 1 | 1 | 1.0000 | 1.0000 |
| 2 | 4 | 8 | 1.4142 | 1.2599 |
| 3 | 9 | 27 | 1.7321 | 1.4422 |
| 4 | 16 | 64 | 2.0000 | 1.5874 |
| 5 | 25 | 125 | 2.2361 | 1.7100 |
| 6 | 36 | 216 | 2.4495 | 1.8171 |
| 7 | 49 | 343 | 2.6458 | 1.9129 |
| 8 | 64 | 512 | 2.8284 | 2.0000 |
| 9 | 81 | 729 | 3.0000 | 2.0801 |
| 10 | 1 00 | 1 000 | 3.1623 | 2.1544 |
| 11 | 1 21 | 1 331 | 3.3166 | 2.2240 |
| 12 | 1 44 | 1 728 | 3.4641 | 2.2894 |
| 13 | 1 69 | 2 197 | 3.6056 | 2.3513 |
| 14 | 1 96 | 2 744 | 3.7417 | 2.4101 |
| 15 | 2 25 | 3 375 | 3.8730 | 2.4662 |
| 16 | 2 56 | 4 096 | 4.0000 | 2.5198 |
| 17 | 2 89 | 4 913 | 4.1231 | 2.5713 |
| 18 | 3 24 | 5 832 | 4.2426 | 2.6207 |
| 19 | 3 61 | 6 859 | 4.3589 | 2.6684 |
| 20 | 4 00 | 8 000 | 4.4721 | 2.7144 |
| 21 | 4 41 | 9 261 | 4.5826 | 2.7589 |
| 22 | 4 84 | 10 648 | 4.6904 | 2.8020 |
| 23 | 5 29 | 12 167 | 4.7958 | 2.8439 |
| 24 | 5 76 | 13 824 | 4.8990 | 2.8845 |
| 25 | 6 25 | 15 625 | 5.0000 | 2.9240 |
| 26 | 6 76 | 17 576 | 5.0990 | 2.9625 |
| 27 | 7 29 | 19 683 | 5.1962 | 3.0000 |
| 28 | 7 84 | 21 952 | 5.2915 | 3.0366 |
| 29 | 8 41 | 24 389 | 5.3852 | 3.0723 |
| 30 | 9 00 | 27 000 | 5.4772 | 3.1072 |
| 31 | 9 61 | 29 791 | 5.5678 | 3.1414 |
| 32 | 10 24 | 32 768 | 5.6569 | 3.1748 |
| 33 | 10 89 | 35 937 | 5.7446 | 3.2075 |
| 34 | 11 56 | 39 304 | 5.8310 | 3.2396 |
| 35 | 12 25 | 42 875 | 5.9161 | 3.2711 |
| 36 | 12 96 | 46 656 | 6.0000 | 3.3019 |
| 37 | 13 69 | 50 653 | 6.0828 | 3.3322 |
| 38 | 14 44 | 54 872 | 6.1644 | 3.3620 |
| 39 | 15 21 | 59 319 | 6.2450 | 3.3912 |
| 40 | 16 00 | 64 000 | 6.3246 | 3.4200 |
| 41 | 16 81 | 68 921 | 6.4031 | 3.4482 |
| 42 | 17 64 | 74 088 | 6.4807 | 3.4760 |
| 43 | 18 49 | 79 507 | 6.5574 | 3.5034 |
| 44 | 19 36 | 85 184 | 6.6332 | 3.5303 |
| 45 | 20 25 | 91 125 | 6.7082 | 3.5569 |
| 46 | 21 16 | 97 336 | 6.7823 | 3.5830 |
| 47 | 22 09 | 103 823 | 6.8557 | 3.6088 |
| 48 | 23 04 | 110 592 | 6.9282 | 3.6342 |
| 49 | 24 01 | 117 649 | 7.0000 | 3.6593 |
| 50 | 25 00 | 125 000 | 7.0711 | 3.6840 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 51 | 26 01 | 132 651 | 7.1414 | 3.7084 |
| 52 | 27 04 | 140 608 | 7.2111 | 3.7325 |
| 53 | 28 09 | 148 877 | 7.2801 | 3.7563 |
| 54 | 29 16 | 157 464 | 7.3485 | 3 7798 |
| 55 | 30 25 | 166 375 | 7.4162 | 3.8030 |
| 56 | 31 36 | 175 616 | 7.4833 | 3.8259 |
| 57 | 32 49 | 185 193 | 7.5498 | 3.8485 |
| 58 | 33 64 | 195 112 | 7.6158 | 3.8709 |
| 59 | 34 81 | 205 379 | 7.6811 | 3.8930 |
| 60 | 36 00 | 216 000 | 7.7460 | 3.9149 |
| 61 | 37 21 | 226 981 | 7.8102 | 3.9365 |
| 62 | 38 44 | 238 328 | 7.8740 | 3.9579 |
| 63 | 39 69 | 250 047 | 7.9373 | 3.9791 |
| 64 | 40 96 | 262 144 | 8.0000 | 4.0000 |
| 65 | 42 25 | 274 625 | 8.0623 | 4.0207 |
| 66 | 43 56 | 287 496 | 8.1240 | 4.0412 |
| 67 | 44 89 | 300 763 | 8.1854 | 4.0615 |
| 68 | 46 24 | 314 432 | 8.2462 | 4.0817 |
| 69 | 47 61 | 328 509 | 8.3066 | 4.1016 |
| 70 | 49 00 | 343 000 | 8.3666 | 4.1213 |
| 71 | 50 41 | 357 911 | 8.4261 | 4.1408 |
| 72 | 51 84 | 373 248 | 8.4853 | 4.1602 |
| 73 | 53 29 | 389 017 | 8.5440 | 4.1793 |
| 74 | 54 76 | 405 224 | 8.6023 | 4.1983 |
| 75 | 56 25 | 421 875 | 8.6603 | 4.2172 |
| 76 | 57 76 | 438 976 | 8.7178 | 4.2358 |
| 77 | 59 29 | 456 533 | 8.7750 | 4.2543 |
| 78 | 60 84 | 474 552 | 8.8318 | 4.2727 |
| 79 | 62 41 | 493 039 | 8.8882 | 4.2908 |
| 80 | 64 00 | 512 000 | 8.9443 | 4.3089 |
| 81 | 65 61 | 531 441 | 9.0000 | 4.3267 |
| 82 | 67 24 | 551 368 | 9.0554 | 4.3445 |
| 83 | 68 89 | 571 787 | 9.1104 | 4.3621 |
| 84 | 70 56 | 592 704 | 9.1652 | 4.3795 |
| 85 | 72 25 | 614 125 | 9.2195 | 4.3968 |
| 86 | 73 96 | 636 056 | 9.2736 | 4.4140 |
| 87 | 75 69 | 658 503 | 9.3274 | 4.4310 |
| 88 | 77 44 | 681 472 | 9.3808 | 4.4480 |
| 89 | 79 21 | 704 969 | 9.4340 | 4.4647 |
| 90 | 81 00 | 729 000 | 9.4868 | 4.4814 |
| 91 | 82 81 | 753 571 | 9.5394 | 4.4979 |
| 92 | 84 64 | 778 688 | 9.5917 | 4.5144 |
| 93 | 86 49 | 804 357 | 9.6437 | 4.5307 |
| 94 | 88 36 | 830 584 | 9.6954 | 4.5468 |
| 95 | 90 25 | 857 375 | 9.7468 | 4.5629 |
| 96 | 92 16 | 884 736 | 9.7980 | 4.5789 |
| 97 | 94 09 | 912 673 | 9.8489 | 4.5947 |
| 98 | 96 04 | 941 192 | 9.8995 | 4.6104 |
| 99 | 98 01 | 970 299 | 9.9499 | 4.6261 |
| 100 | 1 00 00 | 1 000 000 | 10.0000 | 4.6416 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 101 | 1 02 01 | 1 030 301 | 10.0499 | 4.6570 |
| 102 | 1 04 04 | 1 061 208 | 10.0995 | 4.6723 |
| 103 | 1 06 09 | 1 092 727 | 10.1489 | 4.6875 |
| 104 | 1 08 16 | 1 124 864 | 10.1980 | 4.7027 |
| 105 | 1 10 25 | 1 157 625 | 10.2470 | 4.7177 |
| 106 | 1 12 36 | 1 191 016 | 10.2956 | 4.7326 |
| 107 | 1 14 49 | 1 225 043 | 10.3441 | 4.7475 |
| 108 | 1 16 64 | 1 259 712 | 10.3923 | 4.7622 |
| 109 | 1 18 81 | 1 295 029 | 10.4403 | 4.7769 |
| 110 | 1 21 00 | 1 331 000 | 10.4881 | 4.7914 |
| 111 | 1 23 21 | 1 367 631 | 10.5357 | 4.8059 |
| 112 | 1 25 44 | 1 404 928 | 10.5830 | 4.8203 |
| 113 | 1 27 69 | 1 442 897 | 10.6301 | 4.8346 |
| 114 | 1 29 96 | 1 481 544 | 10.6771 | 4.8488 |
| 115 | 1 32 25 | 1 520 875 | 10.7238 | 4.8629 |
| 116 | 1 34 56 | 1 560 896 | 10.7703 | 4.8770 |
| 117 | 1 36 89 | 1 601 613 | 10.8167 | 4.8910 |
| 118 | 1 39 24 | 1 643 032 | 10.8628 | 4.9049 |
| 119 | 1 41 61 | 1 685 159 | 10.9087 | 4.9187 |
| 120 | 1 44 00 | 1 728 000 | 10.9545 | 4.9324 |
| 121 | 1 46 41 | 1 771 561 | 11.0000 | 4.9461 |
| 122 | 1 48 84 | 1 815 848 | 11.0454 | 4.9597 |
| 123 | 1 51 29 | 1 860 867 | 11.0905 | 4.9732 |
| 124 | 1 53 76 | 1 906 624 | 11.1355 | 4.9866 |
| 125 | 1 56 25 | 1 953 125 | 11.1803 | 5.0000 |
| 126 | 1 58 76 | 2 000 376 | 11.2250 | 5.0133 |
| 127 | 1 61 29 | 2 048 383 | 11.2694 | 5.0265 |
| 128 | 1 63 84 | 2 097 152 | 11.3137 | 5.0397 |
| 129 | 1 66 41 | 2 146 689 | 11.3578 | 5.0528 |
| 130 | 1 69 00 | 2 197 000 | 11.4018 | 5.0658 |
| 131 | 1 71 61 | 2 248 091 | 11.4455 | 5.0788 |
| 132 | 1 74 24 | 2 299 968 | 11.4891 | 5.0916 |
| 133 | 1 76 89 | 2 352 637 | 11.5326 | 5.1045 |
| 134 | 1 79 56 | 2 406 104 | 11.5758 | 5.1172 |
| 135 | 1 82 25 | 2 460 375 | 11.6190 | 5.1299 |
| 136 | 1 84 96 | 2 515 456 | 11.6619 | 5.1426 |
| 137 | 1 87 69 | 2 571 353 | 11.7047 | 5.1551 |
| 138 | 1 90 44 | 2 628 072 | 11.7473 | 5.1676 |
| 139 | 1 93 21 | 2 685 619 | 11.7898 | 5.1801 |
| 140 | 1 96 00 | 2 744 000 | 11.8322 | 5.1925 |
| 141 | 1 98 81 | 2 803 221 | 11.8743 | 5.2048 |
| 142 | 2 01 64 | 2 863 288 | 11.9164 | 5.2171 |
| 143 | 2 04 49 | 2 924 207 | 11.9583 | 5.2293 |
| 144 | 2 07 36 | 2 985 984 | 12.0000 | 5.2415 |
| 145 | 2 10 25 | 3 048 625 | 12.0416 | 5.2536 |
| 146 | 2 13 16 | 3 112 136 | 12.0830 | 5.2656 |
| 147 | 2 16 09 | 3 176 523 | 12.1244 | 5.2776 |
| 148 | 2 19 04 | 3 241 792 | 12.1655 | 5.2896 |
| 149 | 2 22 01 | 3 307 949 | 12.2066 | 5.3015 |
| 150 | 2 25 00 | 3 375 000 | 12.2474 | 5.3133 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 151 | 2 28 01 | 3 442 951 | 12.2882 | 5.3251 |
| 152 | 2 31 04 | 3 511 808 | 12.3288 | 5.3368 |
| 153 | 2 34 09 | 3 581 577 | 12.3693 | 5.3485 |
| 154 | 2 37 16 | 3 652 264 | 12.4097 | 5.3601 |
| 155 | 2 40 25 | 3 723 875 | 12.4499 | 5.3717 |
| 156 | 2 43 36 | 3 796 416 | 12.4900 | 5.3832 |
| 157 | 2 46 49 | 3 869 893 | 12.5300 | 5.3947 |
| 158 | 2 49 64 | 3 944 312 | 12.5698 | 5.4061 |
| 159 | 2 52 81 | 4 019 679 | 12.6095 | 5.4175 |
| 160 | 2 56 00 | 4 096 000 | 12.6491 | 5.4288 |
| 161 | 2 59 21 | 4 173 281 | 12.6886 | 5.4401 |
| 162 | 2 62 44 | 4 251 528 | 12.7279 | 5.4514 |
| 163 | 2 65 69 | 4 330 747 | 12.7671 | 5.4626 |
| 164 | 2 68 96 | 4 410 944 | 12.8062 | 5.4737 |
| 165 | 2 72 25 | 4 492 125 | 12.8452 | 5.4848 |
| 166 | 2 75 56 | 4 574 296 | 12.8841 | 5.4959 |
| 167 | 2 78 89 | 4 657 463 | 12.9228 | 5.5069 |
| 168 | 2 82 24 | 4 741 632 | 12.9615 | 5.5178 |
| 169 | 2 85 61 | 4 826 809 | 13.0000 | 5.5288 |
| 170 | 2 89 00 | 4 913 000 | 13.0384 | 5.5397 |
| 171 | 2 92 41 | 5 000 211 | 13.0767 | 5.5505 |
| 172 | 2 95 84 | 5 088 448 | 13.1149 | 5.5613 |
| 173 | 2 99 29 | 5 177 717 | 13.1529 | 5.5721 |
| 174 | 3 02 76 | 5 268 024 | 13.1909 | 5.5828 |
| 175 | 3 06 25 | 5 359 375 | 13.2288 | 5.5934 |
| 176 | 3 09 76 | 5 451 776 | 13.2665 | 5.6041 |
| 177 | 3 13 29 | 5 545 233 | 13.3041 | 5.6147 |
| 178 | 3 16 84 | 5 639 752 | 13.3417 | 5.6252 |
| 179 | 3 20 41 | 5 735 339 | 13.3791 | 5.6357 |
| 180 | 3 24 00 | 5 832 000 | 13.4164 | 5.6462 |
| 181 | 3 27 61 | 5 929 741 | 13.4536 | 5.6567 |
| 182 | 3 31 24 | 6 028 568 | 13.4907 | 5.6671 |
| 183 | 3 34 89 | 6 128 487 | 13.5277 | 5.6774 |
| 184 | 3 38 56 | 6 229 504 | 13.5647 | 5.6877 |
| 185 | 3 42 25 | 6 331 625 | 13.6015 | 5.6980 |
| 186 | 3 45 96 | 6 434 856 | 13.6382 | 5.7083 |
| 187 | 3 49 69 | 6 539 203 | 13.6748 | 5.7185 |
| 188 | 3 53 44 | 6 644 672 | 13.7113 | 5.7287 |
| 189 | 3 57 21 | 6 751 269 | 13.7477 | 5.7388 |
| 190 | 3 61 00 | 6 859 000 | 13.7840 | 5.7489 |
| 191 | 3 64 81 | 6 967 871 | 13.8203 | 5.7590 |
| 192 | 3 68 64 | 7 077 888 | 13.8564 | 5.7690 |
| 193 | 3 72 49 | 7 189 057 | 13.8924 | 5.7790 |
| 194 | 3 76 36 | 7 301 384 | 13.9284 | 5.7890 |
| 195 | 3 80 25 | 7 414 875 | 13.9642 | 5.7989 |
| 196 | 3 84 16 | 7 529 536 | 14.0000 | 5.8088 |
| 197 | 3 88 09 | 7 645 373 | 14.0357 | 5.8186 |
| 198 | 3 92 04 | 7 762 392 | 14.0712 | 5.8285 |
| 199 | 3 96 01 | 7 880 599 | 14.1067 | 5.8383 |
| 200 | 4 00 00 | 8 000 000 | 14.1421 | 5.8480 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 201 | 4 04 01 | 8 120 601 | 14.1774 | 5.8578 |
| 202 | 4 08 04 | 8 242 408 | 14.2127 | 5.8675 |
| 203 | 4 12 09 | 8 365 427 | 14.2478 | 5.8771 |
| 204 | 4 16 16 | 8 489 664 | 14.2829 | 5.8868 |
| 205 | 4 20 25 | 8 615 125 | 14.3178 | 5.8964 |
| 206 | 4 24 36 | 8 741 816 | 14.3527 | 5.9059 |
| 207 | 4 28 49 | 8 869 743 | 14.3875 | 5.9155 |
| 208 | 4 32 64 | 8 998 912 | 14.4222 | 5.9250 |
| 209 | 4 36 81 | 9 129 329 | 14.4568 | 5.9345 |
| 210 | 4 41 00 | 9 261 000 | 14.4914 | 5.9439 |
| 211 | 4 45 21 | 9 393 931 | 14.5258 | 5.9533 |
| 212 | 4 49 44 | 9 528 128 | 14.5602 | 5.9627 |
| 213 | 4 53 69 | 9 663 597 | 14.5945 | 5.9721 |
| 214 | 4 57 96 | 9 800 344 | 14.6287 | 5.9814 |
| 215 | 4 62 25 | 9 938 375 | 14.6629 | 5.9907 |
| 216 | 4 66 56 | 10 077 696 | 14.6969 | 6.0000 |
| 217 | 4 70 89 | 10 218 313 | 14.7309 | 6.0092 |
| 218 | 4 75 24 | 10 360 232 | 14.7648 | 6.0185 |
| 219 | 4 79 61 | 10 503 459 | 14.7986 | 6.0277 |
| 220 | 4 84 00 | 10 648 000 | 14.8324 | 6.0368 |
| 221 | 4 88 41 | 10 793 861 | 14.8661 | 6.0459 |
| 222 | 4 92 84 | 10 941 048 | 14.8997 | 6.0550 |
| 223 | 4 97 29 | 11 089 567 | 14.9332 | 6.0641 |
| 224 | 5 01 76 | 11 239 424 | 14.9666 | 6.0732 |
| 225 | 5 06 25 | 11 390 625 | 15.0000 | 6.0822 |
| 226 | 5 10 76 | 11 543 176 | 15.0333 | 6.0912 |
| 227 | 5 15 29 | 11 697 083 | 15.0665 | 6.1002 |
| 228 | 5 19 84 | 11 852 352 | 15.0997 | 6.1091 |
| 229 | 5 24 41 | 12 008 989 | 15.1327 | 6.1180 |
| 230 | 5 29 00 | 12 167 000 | 15.1658 | 6.1269 |
| 231 | 5 33 61 | 12 326 391 | 15.1987 | 6.1358 |
| 232 | 5 38 24 | 12 487 168 | 15.2315 | 6.1446 |
| 233 | 5 42 89 | 12 649 337 | 15.2643 | 6.1534 |
| 234 | 5 47 56 | 12 812 904 | 15.2971 | 6.1622 |
| 235 | 5 52 25 | 12 977 875 | 15.3297 | 6.1710 |
| 236 | 5 56 96 | 13 144 256 | 15.3623 | 6.1797 |
| 237 | 5 61 69 | 13 312 053 | 15.3948 | 6.1885 |
| 238 | 5 66 44 | 13 481 272 | 15.4272 | 6.1972 |
| 239 | 5 71 21 | 13 651 919 | 15.4596 | 6.2058 |
| 240 | 5 76 00 | 13 824 000 | 15.4919 | 6.2145 |
| 241 | 5 80 81 | 13 997 521 | 15.5242 | 6.2231 |
| 242 | 5 85 64 | 14 172 488 | 15.5563 | 6.2317 |
| 243 | 5 90 49 | 14 348 907 | 15.5885 | 6.2403 |
| 244 | 5 95 36 | 14 526 784 | 15.6205 | 6.2488 |
| 245 | 6 00 25 | 14 706 125 | 15.6525 | 6.2573 |
| 246 | 6 05 16 | 14 886 936 | 15.6844 | 6.2658 |
| 247 | 6 10 09 | 15 069 223 | 15.7162 | 6.2743 |
| 248 | 6 15 04 | 15 252 992 | 15.7480 | 6.2828 |
| 249 | 6 20 01 | 15 438 249 | 15.7797 | 6.2912 |
| 250 | 6 25 00 | 15 625 000 | 15.8114 | 6.2996 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 251 | 6 30 01 | 15 813 251 | 15.8430 | 6.3080 |
| 252 | 6 35 04 | 16 003 008 | 15.8745 | 6.3164 |
| 253 | 6 40 09 | 16 194 277 | 15.9060 | 6.3247 |
| 254 | 6 45 16 | 16 387 064 | 15.9374 | 6.3330 |
| 255 | 6 50 25 | 16 581 375 | 15.9687 | 6.3413 |
| 256 | 6 55 36 | 16 777 216 | 16.0000 | 6.3496 |
| 257 | 6 60 49 | 16 974 593 | 16.0312 | 6.3579 |
| 258 | 6 65 64 | 17 173 512 | 16.0624 | 6.3661 |
| 259 | 6 70 81 | 17 373 979 | 16.0935 | 6.3743 |
| 260 | 6 76 00 | 17 576 000 | 16.1245 | 6.3825 |
| 261 | 6 81 21 | 17 779 581 | 16.1555 | 6.3907 |
| 262 | 6 86 44 | 17 984 728 | 16.1864 | 6.3988 |
| 263 | 6 91 69 | 18 191 447 | 16.2173 | 6.4070 |
| 264 | 6 96 96 | 18 399 744 | 16.2481 | 6.4151 |
| 265 | 7 02 25 | 18 609 625 | 16.2788 | 6.4232 |
| 266 | 7 07 56 | 18 821 096 | 16.3095 | 6.4312 |
| 267 | 7 12 89 | 19 034 163 | 16.3401 | 6.4393 |
| 268 | 7 18 24 | 19 248 832 | 16.3707 | 6.4473 |
| 269 | 7 23 61 | 19 465 109 | 16.4012 | 6.4553 |
| 270 | 7 29 00 | 19 683 000 | 16.4317 | 6.4633 |
| 271 | 7 34 41 | 19 902 511 | 16.4621 | 6.4713 |
| 272 | 7 39 84 | 20 123 648 | 16.4924 | 6.4792 |
| 273 | 7 45 29 | 20 346 417 | 16.5227 | 6.4872 |
| 274 | 7 50 76 | 20 570 824 | 16.5529 | 6.4951 |
| 275 | 7 56 25 | 20 796 875 | 16.5831 | 6.5030 |
| 276 | 7 61 76 | 21 024 576 | 16.6132 | 6.5108 |
| 277 | 7 67 29 | 21 253 933 | 16.6433 | 6.5187 |
| 278 | 7 72 84 | 21 484 952 | 16.6733 | 6.5265 |
| 279 | 7 78 41 | 21 717 639 | 16.7033 | 6.5343 |
| 280 | 7 84 00 | 21 952 000 | 16.7332 | 6.5421 |
| 281 | 7 89 61 | 22 188 041 | 16.7631 | 6.5499 |
| 282 | 7 95 24 | 22 425 768 | 16.7929 | 6.5577 |
| 283 | 8 00 89 | 22 665 187 | 16.8226 | 6.5654 |
| 284 | 8 06 56 | 22 906 304 | 16.8523 | 6.5731 |
| 285 | 8 12 25 | 23 149 125 | 16.8819 | 6.5808 |
| 286 | 8 17 96 | 23 393 656 | 16.9115 | 6.5885 |
| 287 | 8 23 69 | 23 639 903 | 16.9411 | 6.5962 |
| 288 | 8 29 44 | 23 887 872 | 16.9706 | 6.6039 |
| 289 | 8 35 21 | 24 137 569 | 17.0000 | 6.6115 |
| 290 | 8 41 00 | 24 389 000 | 17.0294 | 6.6191 |
| 291 | 8 46 81 | 24 642 171 | 17.0587 | 6.6267 |
| 292 | 8 52 64 | 24 897 088 | 17.0880 | 6.6343 |
| 293 | 8 58 49 | 25 153 757 | 17.1172 | 6.6419 |
| 294 | 8 64 36 | 25 412 184 | 17.1464 | 6.6494 |
| 295 | 8 70 25 | 25 672 375 | 17.1756 | 6.6569 |
| 296 | 8 76 16 | 25 934 336 | 17.2047 | 6.6644 |
| 297 | 8 82 09 | 26 198 073 | 17.2337 | 6.6719 |
| 298 | 8 88 04 | 26 463 592 | 17.2627 | 6.6794 |
| 299 | 8 94 01 | 26 730 899 | 17.2916 | 6.6869 |
| 300 | 9 00 00 | 27 000 000 | 17.3205 | 6.6943 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 301 | 9 06 01 | 27 270 901 | 17.3494 | 6.7018 |
| 302 | 9 12 04 | 27 543 608 | 17.3781 | 6.7092 |
| 303 | 9 18 09 | 27 818 127 | 17.4069 | 6.7166 |
| 304 | 9 24 16 | 28 094 464 | 17.4356 | 6.7240 |
| 305 | 9 30 25 | 28 372 625 | 17.4642 | 6.7313 |
| 306 | 9 36 36 | 28 652 616 | 17.4929 | 6.7387 |
| 307 | 9 42 49 | 28 934 443 | 17.5214 | 6.7460 |
| 308 | 9 48 64 | 29 218 112 | 17.5499 | 6.7533 |
| 309 | 9 54 81 | 29 503 629 | 17.5784 | 6.7606 |
| 310 | 9 61 00 | 29 791 000 | 17.6068 | 6.7679 |
| 311 | 9 67 21 | 30 080 231 | 17.6352 | 6.7752 |
| 312 | 9 73 44 | 30 371 328 | 17.6635 | 6.7824 |
| 313 | 9 79 69 | 30 664 297 | 17.6918 | 6.7897 |
| 314 | 9 85 96 | 30 959 144 | 17.7200 | 6.7969 |
| 315 | 9 92 25 | 31 255 875 | 17.7482 | 6.8041 |
| 316 | 9 98 56 | 31 554 496 | 17.7764 | 6.8113 |
| 317 | 10 04 89 | 31 855 013 | 17.8045 | 6.8185 |
| 318 | 10 11 24 | 32 157 432 | 17.8326 | 6.8256 |
| 319 | 10 17 61 | 32 461 759 | 17.8606 | 6.8328 |
| 320 | 10 24 00 | 32 768 000 | 17.8885 | 6.8399 |
| 321 | 10 30 41 | 33 076 161 | 17.9165 | 6.8470 |
| 322 | 10 36 84 | 33 386 248 | 17.9444 | 6.8541 |
| 323 | 10 43 29 | 33 698 267 | 17.9722 | 6.8612 |
| 324 | 10 49 76 | 34 012 224 | 18.0000 | 6.8683 |
| 325 | 10 56 25 | 34 328 125 | 18.0278 | 6.8753 |
| 326 | 10 62 76 | 34 645 976 | 18.0555 | 6.8824 |
| 327 | 10 69 29 | 34 965 783 | 18.0831 | 6.8894 |
| 328 | 10 75 84 | 35 287 552 | 18.1108 | 6.8964 |
| 329 | 10 82 41 | 35 611 289 | 18.1384 | 6.9034 |
| 330 | 10 89 00 | 35 937 000 | 18.1659 | 6.9104 |
| 331 | 10 95 61 | 36 264 691 | 18.1934 | 6.9174 |
| 332 | 11 02 24 | 36 594 368 | 18.2209 | 6.9244 |
| 333 | 11 08 89 | 36 926 037 | 18.2483 | 6.9313 |
| 334 | 11 15 56 | 37 259 704 | 18.2757 | 6.9382 |
| 335 | 11 22 25 | 37 595 375 | 18.3030 | 6.9451 |
| 336 | 11 28 96 | 37 933 056 | 18.3303 | 6.9521 |
| 337 | 11 35 69 | 38 272 753 | 18.3576 | 6.9589 |
| 338 | 11 42 44 | 38 614 472 | 18.3848 | 6.9658 |
| 339 | 11 49 21 | 38 958 219 | 18.4120 | 6.9727 |
| 340 | 11 56 00 | 39 304 000 | 18.4391 | 6.9795 |
| 341 | 11 62 81 | 39 651 821 | 18.4662 | 6.9864 |
| 342 | 11 69 64 | 40 001 688 | 18.4932 | 6.9932 |
| 343 | 11 76 49 | 40 353 607 | 18.5203 | 7.0000 |
| 344 | 11 83 36 | 40 707 584 | 18.5472 | 7.0068 |
| 345 | 11 90 25 | 41 063 625 | 18.5742 | 7.0136 |
| 346 | 11 97 16 | 41 421 736 | 18.6011 | 7.0203 |
| 347 | 12 04 09 | 41 781 923 | 18.6279 | 7.0271 |
| 348 | 12 11 04 | 42 144 192 | 18.6548 | 7.0338 |
| 349 | 12 18 01 | 42 508 549 | 18.6815 | 7.0406 |
| 350 | 12 25 00 | 42 875 000 | 18.7083 | 7.0473 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 351 | 12 32 01 | 43 243 551 | 18.7350 | 7.0540 |
| 352 | 12 39 04 | 43 614 208 | 18.7617 | 7.0607 |
| 353 | 12 46 09 | 43 986 977 | 18.7883 | 7.0674 |
| 354 | 12 53 16 | 44 361 864 | 18.8149 | 7.0740 |
| 355 | 12 60 25 | 44 738 875 | 18.8414 | 7.0807 |
| 356 | 12 67 36 | 45 118 016 | 18.8680 | 7.0873 |
| 357 | 12 74 49 | 45 499 293 | 18.8944 | 7.0940 |
| 358 | 12 81 64 | 45 882 712 | 18.9209 | 7.1006 |
| 359 | 12 88 81 | 46 268 279 | 18.9473 | 7.1072 |
| 360 | 12 96 00 | 46 656 000 | 18.9737 | 7.1138 |
| 361 | 13 03 21 | 47 045 881 | 19.0000 | 7.1204 |
| 362 | 13 10 44 | 47 437 928 | 19.0263 | 7.1269 |
| 363 | 13 17 69 | 47 832 147 | 19.0526 | 7.1335 |
| 364 | 13 24 96 | 48 228 544 | 19.0788 | 7.1400 |
| 365 | 13 32 25 | 48 627 125 | 19.1050 | 7.1466 |
| 366 | 13 39 56 | 49 027 896 | 19.1311 | 7.1531 |
| 367 | 13 46 89 | 49 430 863 | 19.1572 | 7.1596 |
| 368 | 13 54 24 | 49 836 032 | 19.1833 | 7.1661 |
| 369 | 13 61 61 | 50 243 409 | 19.2094 | 7.1726 |
| 370 | 13 69 00 | 50 653 000 | 19.2354 | 7.1791 |
| 371 | 13 76 41 | 51 064 811 | 19.2614 | 7.1855 |
| 372 | 13 83 84 | 51 478 848 | 19.2873 | 7.1920 |
| 373 | 13 91 29 | 51 895 117 | 19.3132 | 7.1984 |
| 374 | 13 98 76 | 52 313 624 | 19.3391 | 7.2048 |
| 375 | 14 06 25 | 52 734 375 | 19.3649 | 7.2112 |
| 376 | 14 13 76 | 53 157 376 | 19.3907 | 7.2177 |
| 377 | 14 21 29 | 53 582 633 | 19.4165 | 7.2240 |
| 378 | 14 28 84 | 54 010 152 | 19.4422 | 7.2304 |
| 379 | 14 36 41 | 54 439 939 | 19.4679 | 7.2368 |
| 380 | 14 44 00 | 54 872 000 | 19.4936 | 7.2432 |
| 381 | 14 51 61 | 55 306 341 | 19.5192 | 7.2495 |
| 382 | 14 59 24 | 55 742 968 | 19.5448 | 7.2558 |
| 383 | 14 66 89 | 56 181 887 | 19.5704 | 7.2622 |
| 384 | 14 74 56 | 56 623 104 | 19.5959 | 7.2685 |
| 385 | 14 82 25 | 57 066 625 | 19.6214 | 7.2748 |
| 386 | 14 89 96 | 57 512 456 | 19.6469 | 7.2811 |
| 387 | 14 97 69 | 57 960 603 | 19.6723 | 7.2874 |
| 388 | 15 05 44 | 58 411 072 | 19.6977 | 7.2936 |
| 389 | 15 13 21 | 58 863 869 | 19.7231 | 7.2999 |
| 390 | 15 21 00 | 59 319 000 | 19.7484 | 7.3061 |
| 391 | 15 28 81 | 59 776 471 | 19.7737 | 7.3124 |
| 392 | 15 36 64 | 60 236 288 | 19.7990 | 7.3186 |
| 393 | 15 44 49 | 60 698 457 | 19.8242 | 7.3248 |
| 394 | 15 52 36 | 61 162 984 | 19.8494 | 7.3310 |
| 395 | 15 60 25 | 61 629 875 | 19.8746 | 7.3372 |
| 396 | 15 68 16 | 62 099 136 | 19.8997 | 7.3434 |
| 397 | 15 76 09 | 62 570 773 | 19.9249 | 7.3496 |
| 398 | 15 84 04 | 63 044 792 | 19.9499 | 7.3558 |
| 399 | 15 92 01 | 63 521 199 | 19.9750 | 7.3619 |
| 400 | 16 00 00 | 64 000 000 | 20.0000 | 7.3681 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 401 | 16 08 01 | 64 481 201 | 20.0250 | 7.3742 |
| 402 | 16 16 04 | 64 964 808 | 20.0499 | 7.3803 |
| 403 | 16 24 09 | 65 450 827 | 20.0749 | 7.3864 |
| 404 | 16 32 16 | 65 939 264 | 20.0998 | 7.3925 |
| 405 | 16 40 25 | 66 430 125 | 20.1246 | 7.3986 |
| 406 | 16 48 36 | 66 923 416 | 20.1494 | 7.4047 |
| 407 | 16 56 49 | 67 419 143 | 20.1742 | 7.4108 |
| 408 | 16 64 64 | 67 917 312 | 20.1990 | 7.4169 |
| 409 | 16 72 81 | 68 417 929 | 20.2237 | 7.4229 |
| 410 | 16 81 00 | 68 921 000 | 20.2485 | 7.4290 |
| 411 | 16 89 21 | 69 426 531 | 20.2731 | 7.4350 |
| 412 | 16 97 44 | 69 934 528 | 20.2978 | 7.4410 |
| 413 | 17 05 69 | 70 444 997 | 20.3224 | 7.4470 |
| 414 | 17 13 96 | 70 957 944 | 20.3470 | 7.4530 |
| 415 | 17 22 25 | 71 473 375 | 20.3715 | 7.4590 |
| 416 | 17 30 56 | 71 991 296 | 20.3961 | 7.4650 |
| 417 | 17 38 89 | 72 511 713 | 20.4206 | 7.4710 |
| 418 | 17 47 24 | 73 034 632 | 20.4450 | 7.4770 |
| 419 | 17 55 61 | 73 560 059 | 20.4695 | 7.4829 |
| 420 | 17 64 00 | 74 088 000 | 20.4939 | 7.4889 |
| 421 | 17 72 41 | 74 618 461 | 20.5183 | 7.4948 |
| 422 | 17 80 84 | 75 151 448 | 20.5426 | 7.5007 |
| 423 | 17 89 29 | 75 686 967 | 20.5670 | 7.5067 |
| 424 | 17 97 76 | 76 225 024 | 20.5913 | 7.5126 |
| 425 | 18 06 25 | 76 765 625 | 20.6155 | 7.5185 |
| 426 | 18 14 76 | 77 308 776 | 20.6398 | 7.5244 |
| 427 | 18 23 29 | 77 854 483 | 20.6640 | 7.5302 |
| 428 | 18 31 84 | 78 402 752 | 20.6882 | 7.5361 |
| 429 | 18 40 41 | 78 953 589 | 20.7123 | 7.5420 |
| 430 | 18 49 00 | 79 507 000 | 20.7364 | 7.5478 |
| 431 | 18 57 61 | 80 062 991 | 20.7605 | 7.5537 |
| 432 | 18 66 24 | 80 621 568 | 20.7846 | 7.5595 |
| 433 | 18 74 89 | 81 182 737 | 20.8087 | 7.5654 |
| 434 | 18 83 56 | 81 746 504 | 20.8327 | 7.5712 |
| 435 | 18 92 25 | 82 312 875 | 20.8567 | 7.5770 |
| 436 | 19 00 96 | 82 881 856 | 20.8806 | 7.5828 |
| 437 | 19 09 69 | 83 453 453 | 20.9045 | 7.5886 |
| 438 | 19 18 44 | 84 027 672 | 20.9284 | 7.5944 |
| 439 | 19 27 21 | 84 604 519 | 20.9523 | 7.6001 |
| 440 | 19 36 00 | 85 184 000 | 20.9762 | 7.6059 |
| 441 | 19 44 81 | 85 766 121 | 21.0000 | 7.6117 |
| 442 | 19 53 64 | 86 350 888 | 21.0238 | 7.6174 |
| 443 | 19 62 49 | 86 938 307 | 21.0476 | 7.6232 |
| 444 | 19 71 36 | 87 528 384 | 21.0713 | 7.6289 |
| 445 | 19 80 25 | 88 121 125 | 21.0950 | 7.6346 |
| 446 | 19 89 16 | 88 716 536 | 21.1187 | 7.6403 |
| 447 | 19 98 09 | 89 314 623 | 21.1424 | 7.6460 |
| 448 | 20 07 04 | 89 915 392 | 21.1660 | 7.6517 |
| 449 | 20 16 01 | 90 518 849 | 21.1896 | 7.6574 |
| 450 | 20 25 00 | 91 125 000 | 21.2132 | 7.6631 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 451 | 20 34 01 | 91 733 851 | 21.2368 | 7.6688 |
| 452 | 20 43 04 | 92 345 408 | 21.2603 | 7.6744 |
| 453 | 20 52 09 | 92 959 677 | 21.2838 | 7.6801 |
| 454 | 20 61 16 | 93 576 664 | 21.3073 | 7.6857 |
| 455 | 20 70 25 | 94 196 375 | 21.3307 | 7.6914 |
| 456 | 20 79 36 | 94 818 816 | 21.3542 | 7.6970 |
| 457 | 20 88 49 | 95 443 993 | 21.3776 | 7.7026 |
| 458 | 20 97 64 | 96 071 912 | 21.4009 | 7.7082 |
| 459 | 21 06 81 | 96 702 579 | 21.4243 | 7.7138 |
| 460 | 21 16 00 | 97 336 000 | 21.4476 | 7.7194 |
| 461 | 21 25 21 | 97 972 181 | 21.4709 | 7.7250 |
| 462 | 21 34 44 | 98 611 128 | 21.4942 | 7.7306 |
| 463 | 21 43 69 | 99 252 847 | 21.5174 | 7.7362 |
| 464 | 21 52 96 | 99 897 344 | 21.5407 | 7.7418 |
| 465 | 21 62 25 | 100 544 625 | 21.5639 | 7.7473 |
| 466 | 21 71 56 | 101 194 696 | 21.5870 | 7.7529 |
| 467 | 21 80 89 | 101 847 563 | 21.6102 | 7.7584 |
| 468 | 21 90 24 | 102 503 232 | 21.6333 | 7.7639 |
| 469 | 21 99 61 | 103 161 709 | 21.6564 | 7.7695 |
| 470 | 22 09 00 | 103 823 000 | 21.6795 | 7.7750 |
| 471 | 22 18 41 | 104 487 111 | 21.7025 | 7.7805 |
| 472 | 22 27 84 | 105 154 048 | 21.7256 | 7.7860 |
| 473 | 22 37 29 | 105 823 817 | 21.7486 | 7.7915 |
| 474 | 22 46 76 | 106 496 424 | 21.7715 | 7.7970 |
| 475 | 22 56 25 | 107 171 875 | 21.7945 | 7.8025 |
| 476 | 22 65 76 | 107 850 176 | 21.8174 | 7.8079 |
| 477 | 22 75 29 | 108 531 333 | 21.8403 | 7.8134 |
| 478 | 22 84 84 | 109 215 352 | 21.8632 | 7.8188 |
| 479 | 22 94 41 | 109 902 239 | 21.8861 | 7.8243 |
| 480 | 23 04 00 | 110 592 000 | 21.9089 | 7.8297 |
| 481 | 23 13 61 | 111 284 641 | 21.9317 | 7.8352 |
| 482 | 23 23 24 | 111 980 168 | 21.9545 | 7.8406 |
| 483 | 23 32 89 | 112 678 587 | 21.9773 | 7.8460 |
| 484 | 23 42 56 | 113 379 904 | 22.0000 | 7.8514 |
| 485 | 23 52 25 | 114 084 125 | 22.0227 | 7.8568 |
| 486 | 23 61 96 | 114 791 256 | 22.0454 | 7.8622 |
| 487 | 23 71 69 | 115 501 303 | 22.0681 | 7.8676 |
| 488 | 23 81 44 | 116 214 272 | 22.0907 | 7.8730 |
| 489 | 23 91 21 | 116 930 169 | 22.1133 | 7.8784 |
| 490 | 24 01 00 | 117 649 000 | 22.1359 | 7.8837 |
| 491 | 24 10 81 | 118 370 771 | 22.1585 | 7.8891 |
| 492 | 24 20 64 | 119 095 488 | 22.1811 | 7.8944 |
| 493 | 24 30 49 | 119 823 157 | 22.2036 | 7.8998 |
| 494 | 24 40 36 | 120 553 784 | 22.2261 | 7.9051 |
| 495 | 24 50 25 | 121 287 375 | 22.2486 | 7.9105 |
| 496 | 24 60 16 | 122 023 936 | 22.2711 | 7.9158 |
| 497 | 24 70 09 | 122 763 473 | 22.2935 | 7.9211 |
| 498 | 24 80 04 | 123 505 992 | 22.3159 | 7.9264 |
| 499 | 24 90 01 | 124 251 499 | 22.3383 | 7.9317 |
| 500 | 25 00 00 | 125 000 000 | 22.3607 | 7.9370 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 501 | 25 10 01 | 125 751 501 | 22.3830 | 7.9423 |
| 502 | 25 20 04 | 126 506 008 | 22.4054 | 7.9476 |
| 503 | 25 30 09 | 127 263 527 | 22.4277 | 7.9528 |
| 504 | 25 40 16 | 128 024 064 | 22.4499 | 7.9581 |
| 505 | 25 50 25 | 128 787 625 | 22.4722 | 7.9634 |
| 506 | 25 60 36 | 129 554 216 | 22.4944 | 7.9686 |
| 507 | 25 70 49 | 130 323 843 | 22.5167 | 7.9739 |
| 508 | 25 80 63 | 131 096 512 | 22.5389 | 7.9791 |
| 509 | 25 90 81 | 131 872 229 | 22.5610 | 7.9843 |
| 510 | 26 01 00 | 132 651 000 | 22.5832 | 7.9896 |
| 511 | 26 11 21 | 133 432 831 | 22.6053 | 7.9948 |
| 512 | 26 21 44 | 134 217 728 | 22.6274 | 8.0000 |
| 513 | 26 31 69 | 135 005 697 | 22.6495 | 8.0052 |
| 514 | 26 41 96 | 135 796 744 | 22.6716 | 8.0104 |
| 515 | 26 52 25 | 136 590 875 | 22.6936 | 8.0156 |
| 516 | 26 62 56 | 137 388 096 | 22.7156 | 8.0208 |
| 517 | 26 72 89 | 138 188 413 | 22.7376 | 8.0260 |
| 518 | 26 83 24 | 138 991 832 | 22.7596 | 8.0311 |
| 519 | 26 93 61 | 139 798 359 | 22.7816 | 8.0363 |
| 520 | 27 04 00 | 140 608 000 | 22.8035 | 8.0415 |
| 521 | 27 14 41 | 141 420 761 | 22.8254 | 8.0466 |
| 522 | 27 24 84 | 142 236 648 | 22.8473 | 8.0517 |
| 523 | 27 35 29 | 143 055 667 | 22.8692 | 8.0569 |
| 524 | 27 45 76 | 143 877 824 | 22.8910 | 8.0620 |
| 525 | 27 56 25 | 144 703 125 | 22.9129 | 8.0671 |
| 526 | 27 66 76 | 145 531 576 | 22.9347 | 8.0723 |
| 527 | 27 77 29 | 146 363 183 | 22.9565 | 8.0774 |
| 528 | 27 87 84 | 147 197 952 | 22.9783 | 8.0825 |
| 529 | 27 98 41 | 148 035 889 | 23.0000 | 8.0876 |
| 530 | 28 09 00 | 148 877 000 | 23.0217 | 8.0927 |
| 531 | 28 19 61 | 149 721 291 | 23.0434 | 8.0978 |
| 532 | 28 30 24 | 150 568 768 | 23.0651 | 8.1028 |
| 533 | 28 40 89 | 151 419 437 | 23.0868 | 8.1079 |
| 534 | 28 51 56 | 152 273 304 | 23.1084 | 8.1130 |
| 535 | 28 62 25 | 153 130 375 | 23.1301 | 8.1180 |
| 536 | 28 72 96 | 153 990 656 | 23.1517 | 8.1231 |
| 537 | 28 83 69 | 154 854 153 | 23.1733 | 8.1281 |
| 538 | 28 94 44 | 155 720 872 | 23.1948 | 8.1332 |
| 539 | 29 05 21 | 156 590 819 | 23.2164 | 8.1382 |
| 540 | 29 16 00 | 157 464 000 | 23.2379 | 8.1433 |
| 541 | 29 26 81 | 158 340 421 | 23.2594 | 8.1483 |
| 542 | 29 37 64 | 159 220 088 | 23.2809 | 8.1533 |
| 543 | 29 48 49 | 160 103 007 | 23.3024 | 8.1583 |
| 544 | 29 59 36 | 160 989 184 | 23.3238 | 8.1633 |
| 545 | 29 70 25 | 161 878 625 | 23.3452 | 8.1683 |
| 546 | 29 81 16 | 162 771 336 | 23.3666 | 8.1733 |
| 547 | 29 92 09 | 163 667 323 | 23.3880 | 8.1783 |
| 548 | 30 03 04 | 164 566 592 | 23.4094 | 8.1833 |
| 549 | 30 14 01 | 165 469 149 | 23.4307 | 8.1882 |
| 550 | 30 25 00 | 166 375 000 | 23.4521 | 8.1932 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 551 | 30 36 01 | 167 284 151 | 23.4734 | 8.1982 |
| 552 | 30 47 04 | 168 196 608 | 23.4947 | 8.2031 |
| 553 | 30 58 09 | 169 112 377 | 23.5160 | 8.2081 |
| 554 | 30 69 16 | 170 031 464 | 23.5372 | 8.2130 |
| 555 | 30 80 25 | 170 953 875 | 23.5584 | 8.2180 |
| 556 | 30 91 36 | 171 879 616 | 23.5797 | 8.2229 |
| 557 | 31 02 49 | 172 808 693 | 23.6008 | 8.2278 |
| 558 | 31 13 64 | 173 741 112 | 23.6220 | 8.2327 |
| 559 | 31 24 81 | 174 676 879 | 23.6432 | 8.2377 |
| 560 | 31 36 00 | 175 616 000 | 23.6643 | 8.2426 |
| 561 | 31 47 21 | 176 558 481 | 23.6854 | 8.2475 |
| 562 | 31 58 44 | 177 504 328 | 23.7065 | 8.2524 |
| 563 | 31 69 69 | 178 453 547 | 23.7276 | 8.2573 |
| 564 | 31 80 96 | 179 406 144 | 23.7487 | 8.2621 |
| 565 | 31 92 25 | 180 362 125 | 23.7697 | 8.2670 |
| 566 | 32 03 56 | 181 321 496 | 23.7908 | 8.2719 |
| 567 | 32 14 89 | 182 284 263 | 23.8118 | 8.2768 |
| 568 | 32 26 24 | 183 250 432 | 23.8328 | 8.2816 |
| 569 | 32 37 61 | 184 220 009 | 23.8537 | 8.2865 |
| 570 | 32 49 00 | 185 193 000 | 23.8747 | 8.2913 |
| 571 | 32 60 41 | 186 169 411 | 23.8956 | 8.2962 |
| 572 | 32 71 84 | 187 149 248 | 23.9165 | 8.3010 |
| 573 | 32 83 29 | 188 132 517 | 23.9374 | 8.3059 |
| 574 | 32 94 76 | 189 119 224 | 23.9583 | 8.3107 |
| 575 | 33 06 25 | 190 109 375 | 23.9792 | 8.3155 |
| 576 | 33 17 76 | 191 102 976 | 24.0000 | 8.3203 |
| 577 | 33 29 29 | 192 100 033 | 24.0208 | 8.3251 |
| 578 | 33 40 84 | 193 100 552 | 24.0416 | 8.3300 |
| 579 | 33 52 41 | 194 104 539 | 24.0624 | 8.3348 |
| 580 | 33 64 00 | 195 112 000 | 24.0832 | 8.3396 |
| 581 | 33 75 61 | 196 122 941 | 24.1039 | 8.3443 |
| 582 | 33 87 24 | 197 137 368 | 24.1247 | 8.3491 |
| 583 | 33 98 89 | 198 155 287 | 24.1454 | 8.3539 |
| 584 | 34 10 56 | 199 176 704 | 24.1661 | 8.3587 |
| 585 | 34 22 25 | 200 201 625 | 24.1868 | 8.3634 |
| 586 | 34 33 96 | 201 230 056 | 24.2074 | 8.3682 |
| 587 | 34 45 69 | 202 262 003 | 24.2281 | 8.3730 |
| 588 | 34 57 44 | 203 297 472 | 24.2487 | 8.3777 |
| 589 | 34 69 21 | 204 336 469 | 24.2693 | 8.3825 |
| 590 | 34 81 00 | 205 379 000 | 24.2899 | 8.3872 |
| 591 | 34 92 81 | 206 425 071 | 24.3105 | 8.3919 |
| 592 | 35 04 64 | 207 474 688 | 24.3311 | 8.3967 |
| 593 | 35 16 49 | 208 527 857 | 24.3516 | 8.4014 |
| 594 | 35 28 36 | 209 584 584 | 24.3721 | 8.4061 |
| 595 | 35 40 25 | 210 644 875 | 24.3926 | 8.4108 |
| 596 | 35 52 16 | 211 708 736 | 24.4131 | 8.4155 |
| 597 | 35 64 09 | 212 776 173 | 24.4336 | 8.4202 |
| 598 | 35 76 04 | 213 847 192 | 24.4540 | 8.4249 |
| 599 | 35 88 01 | 214 921 799 | 24.4745 | 8.4296 |
| 600 | 36 00 00 | 216 000 000 | 24.4949 | 8.4343 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 601 | 36 12 01 | 217 081 801 | 24.5153 | 8.4390 |
| 602 | 36 24 04 | 218 167 208 | 24.5357 | 8.4437 |
| 603 | 36 36 09 | 219 256 227 | 24.5561 | 8.4484 |
| 604 | 36 48 16 | 220 348 864 | 24.5764 | 8.4530 |
| 605 | 36 60 25 | 221 445 125 | 24.5967 | 8.4577 |
| 606 | 36 72 36 | 222 545 016 | 24.6171 | 8.4623 |
| 607 | 36 84 49 | 223 648 543 | 24.6374 | 8.4670 |
| 608 | 36 96 64 | 224 755 712 | 24.6577 | 8.4716 |
| 609 | 37 08 81 | 225 866 529 | 24.6779 | 8.4763 |
| 610 | 37 21 00 | 226 981 000 | 24.6982 | 8.4809 |
| 611 | 37 33 21 | 228 099 131 | 24.7184 | 8.4856 |
| 612 | 37 45 44 | 229 220 928 | 24.7386 | 8.4902 |
| 613 | 37 57 69 | 230 346 397 | 24.7588 | 8.4948 |
| 614 | 37 69 96 | 231 475 544 | 24.7790 | 8.4994 |
| 615 | 37 82 25 | 232 608 375 | 24.7992 | 8.5040 |
| 616 | 37 94 56 | 233 744 896 | 24.8193 | 8.5086 |
| 617 | 38 06 89 | 234 885 113 | 24.8395 | 8.5132 |
| 618 | 38 19 24 | 236 029 032 | 24.8596 | 8.5178 |
| 619 | 38 31 61 | 237 176 659 | 24.8797 | 8.5224 |
| 620 | 38 44 00 | 238 328 000 | 24.8998 | 8.5270 |
| 621 | 38 56 41 | 239 483 061 | 24.9199 | 8.5316 |
| 622 | 38 68 84 | 240 641 848 | 24.9399 | 8.5362 |
| 623 | 38 81 29 | 241 804 367 | 24.9600 | 8.5408 |
| 624 | 38 93 76 | 242 970 624 | 24.9800 | 8.5453 |
| 625 | 39 06 25 | 244 140 625 | 25.0000 | 8.5499 |
| 626 | 39 18 76 | 245 314 376 | 25.0200 | 8.5544 |
| 627 | 39 31 29 | 246 491 883 | 25.0400 | 8.5590 |
| 628 | 39 43 84 | 247 673 152 | 25.0599 | 8.5635 |
| 629 | 39 56 41 | 248 858 189 | 25.0799 | 8.5681 |
| 630 | 39 69 00 | 250 047 000 | 25.0998 | 8.5726 |
| 631 | 39 81 61 | 251 239 591 | 25.1197 | 8.5772 |
| 632 | 39 94 24 | 252 435 968 | 25.1396 | 8.5817 |
| 633 | 40 06 89 | 253 636 137 | 25.1595 | 8.5862 |
| 634 | 40 19 56 | 254 840 104 | 25.1794 | 8.5907 |
| 635 | 40 32 25 | 256 047 875 | 25.1992 | 8.5952 |
| 636 | 40 44 96 | 257 259 456 | 25.2190 | 8.5997 |
| 637 | 40 57 69 | 258 474 853 | 25.2389 | 8.6043 |
| 638 | 40 70 44 | 259 694 072 | 25.2587 | 8.6088 |
| 639 | 40 83 21 | 260 917 119 | 25.2784 | 8.6132 |
| 640 | 40 96 00 | 262 144 000 | 25.2982 | 8.6177 |
| 641 | 41 08 81 | 263 374 721 | 25.3180 | 8.6222 |
| 642 | 41 21 64 | 264 609 288 | 25.3377 | 8.6267 |
| 643 | 41 34 49 | 265 847 707 | 25.3574 | 8.6312 |
| 644 | 41 47 36 | 267 089 984 | 25.3772 | 8.6357 |
| 645 | 41 60 25 | 268 336 125 | 25.3969 | 8.6401 |
| 646 | 41 73 16 | 269 586 136 | 25.4165 | 8.6446 |
| 647 | 41 86 09 | 270 840 023 | 25.4362 | 8.6490 |
| 648 | 41 99 04 | 272 097 792 | 25.4558 | 8.6535 |
| 649 | 42 12 01 | 273 359 549 | 25.4755 | 8.6579 |
| 650 | 42 25 00 | 274 625 000 | 25.4951 | 8.6624 |

| Number. | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 651 | 42 38 01 | 275 894 451 | 25.5147 | 8.6668 |
| 652 | 42 51 04 | 277 167 808 | 25.5343 | 8.6713 |
| 653 | 42 64 09 | 278 445 077 | 25.5539 | 8.6757 |
| 654 | 42 77 16 | 279 726 264 | 25.5734 | 8.6801 |
| 655 | 42 90 25 | 281 011 375 | 25.5930 | 8.6845 |
| 656 | 43 03 36 | 282 300 416 | 25.6125 | 8.6890 |
| 657 | 43 16 49 | 283 593 393 | 25.6320 | 8.6934 |
| 658 | 43 29 64 | 284 890 312 | 25.6515 | 8.6978 |
| 659 | 43 42 81 | 286 191 179 | 25.6710 | 8.7022 |
| 660 | 43 56 00 | 287 496 000 | 25.6905 | 8.7066 |
| 661 | 43 69 21 | 288 804 781 | 25.7099 | 8.7110 |
| 662 | 43 82 44 | 290 117 528 | 25.7294 | 8.7154 |
| 663 | 43 95 69 | 291 434 247 | 25.7488 | 8.7198 |
| 664 | 44 08 96 | 292 754 944 | 25.7682 | 8.7241 |
| 665 | 44 22 25 | 294 079 625 | 25.7876 | 8.7285 |
| 666 | 44 35 56 | 295 408 296 | 25.8070 | 8.7329 |
| 667 | 44 48 89 | 296 740 963 | 25.8263 | 8.7373 |
| 668 | 44 62 24 | 298 077 632 | 25.8457 | 8.7416 |
| 669 | 44 75 61 | 299 418 309 | 25.8650 | 8.7460 |
| 670 | 44 89 00 | 300 763 000 | 25.8844 | 8.7503 |
| 671 | 45 02 41 | 302 111 711 | 25.9037 | 8.7547 |
| 672 | 45 15 84 | 303 464 448 | 25.9230 | 8.7590 |
| 673 | 45 29 29 | 304 821 217 | 25.9422 | 8.7634 |
| 674 | 45 42 76 | 306 182 024 | 25.9615 | 8.7677 |
| 675 | 45 56 25 | 307 546 875 | 25.9808 | 8.7721 |
| 676 | 45 69 76 | 308 915 776 | 26.0000 | 8.7764 |
| 677 | 45 83 29 | 310 288 733 | 26.0192 | 8.7807 |
| 678 | 45 96 84 | 311 665 752 | 26.0384 | 8.7850 |
| 679 | 46 10 41 | 313 046 839 | 26.0576 | 8.7893 |
| 680 | 46 24 00 | 314 432 000 | 26.0768 | 8.7937 |
| 681 | 46 37 61 | 315 821 241 | 26.0960 | 8.7980 |
| 682 | 46 51 24 | 317 214 568 | 26.1151 | 8.8023 |
| 683 | 46 64 89 | 318 611 987 | 26.1343 | 8.8066 |
| 684 | 46 78 56 | 320 013 504 | 26.1534 | 8.8109 |
| 685 | 46 92 25 | 321 419 125 | 26.1725 | 8.8152 |
| 686 | 47 05 96 | 322 828 856 | 26.1916 | 8.8194 |
| 687 | 47 19 69 | 324 242 703 | 26.2107 | 8.8237 |
| 688 | 47 33 44 | 325 660 672 | 26.2298 | 8.8280 |
| 689 | 47 47 21 | 327 082 769 | 26.2488 | 8.8323 |
| 690 | 47 61 00 | 328 509 000 | 26.2679 | 8.8366 |
| 691 | 47 74 81 | 329 939 371 | 26.2869 | 8.8408 |
| 692 | 47 88 64 | 331 373 888 | 26.3059 | 8.8451 |
| 693 | 48 02 49 | 332 812 557 | 26.3249 | 8.8493 |
| 694 | 48 16 36 | 334 255 384 | 26.3439 | 8.8536 |
| 695 | 48 30 25 | 335 702 375 | 26.3629 | 8.8578 |
| 696 | 48 44 16 | 337 153 536 | 26.3818 | 8.8621 |
| 697 | 48 58 09 | 338 608 873 | 26.4008 | 8.8663 |
| 698 | 48 72 04 | 340 068 392 | 26.4197 | 8.8706 |
| 699 | 48 86 01 | 341 532 099 | 26.4386 | 8.8748 |
| 700 | 49 00 00 | 343 000 000 | 26.4575 | 8.8790 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 701 | 49 14 01 | 344 472 101 | 26.4764 | 8.8833 |
| 702 | 49 28 04 | 345 948 408 | 26.4953 | 8.8875 |
| 703 | 49 42 09 | 347 428 927 | 26.5141 | 8.8917 |
| 704 | 49 56 16 | 348 913 664 | 26.5330 | 8.8959 |
| 705 | 49 70 25 | 350 402 625 | 26.5518 | 8.9001 |
| 706 | 49 84 36 | 351 895 816 | 26.5707 | 8.9043 |
| 707 | 49 98 49 | 353 393 243 | 26.5895 | 8.9085 |
| 708 | 50 12 64 | 354 894 912 | 26.6083 | 8.9127 |
| 709 | 50 26 81 | 356 400 829 | 26.6271 | 8.9169 |
| 710 | 50 41 00 | 357 911 000 | 26.6458 | 8.9211 |
| 711 | 50 55 21 | 359 425 431 | 26.6646 | 8.9253 |
| 712 | 50 69 44 | 360 944 128 | 26.6833 | 8.9295 |
| 713 | 50 83 69 | 362 467 097 | 26.7021 | 8.9337 |
| 714 | 50 97 96 | 363 994 344 | 26.7208 | 8.9378 |
| 715 | 51 12 25 | 365 525 875 | 26.7395 | 8.9420 |
| 716 | 51 26 56 | 367 061 696 | 26.7582 | 8.9462 |
| 717 | 51 40 89 | 368 601 813 | 26.7769 | 8.9503 |
| 718 | 51 55 24 | 370 146 232 | 26.7955 | 8.9545 |
| 719 | 51 69 61 | 371 694 959 | 26.8142 | 8.9587 |
| 720 | 51 84 00 | 373 248 000 | 26.8328 | 8.9628 |
| 721 | 51 98 41 | 374 805 361 | 26.8514 | 8.9670 |
| 722 | 52 12 84 | 376 367 048 | 26.8701 | 8.9711 |
| 723 | 52 27 29 | 377 933 067 | 26.8887 | 8.9752 |
| 724 | 52 41 76 | 379 503 424 | 26.9072 | 8.9794 |
| 725 | 52 56 25 | 381 078 125 | 26.9258 | 8.9835 |
| 726 | 52 70 76 | 382 657 176 | 26.9444 | 8.9876 |
| 727 | 52 85 29 | 384 240 583 | 26.9629 | 8.9918 |
| 728 | 52 99 84 | 385 828 352 | 26.9815 | 8.9959 |
| 729 | 53 14 41 | 387 420 489 | 27.0000 | 9.0000 |
| 730 | 53 29 00 | 389 017 000 | 27.0185 | 9.0041 |
| 731 | 53 43 61 | 390 617 891 | 27.0370 | 9.0082 |
| 732 | 53 58 24 | 392 223 168 | 27.0555 | 9.0123 |
| 733 | 53 72 89 | 393 832 837 | 27.0740 | 9.0164 |
| 734 | 53 87 56 | 395 446 904 | 27.0924 | 9.0205 |
| 735 | 54 02 25 | 397 065 375 | 27.1109 | 9.0246 |
| 736 | 54 16 96 | 398 688 256 | 27.1293 | 9.0287 |
| 737 | 54 31 69 | 400 315 553 | 27.1477 | 9.0328 |
| 738 | 54 46 44 | 401 947 272 | 27.1662 | 9.0369 |
| 739 | 54 61 21 | 403 583 419 | 27.1846 | 9.0410 |
| 740 | 54 76 00 | 405 224 000 | 27.2029 | 9.0450 |
| 741 | 54 90 81 | 406 869 021 | 27.2213 | 9.0491 |
| 742 | 55 05 64 | 408 518 488 | 27.2397 | 9.0532 |
| 743 | 55 20 49 | 410 172 407 | 27.2580 | 9.0572 |
| 744 | 55 35 36 | 411 830 784 | 27.2764 | 9.0613 |
| 745 | 55 50 25 | 413 493 625 | 27.2947 | 9.0654 |
| 746 | 55 65 16 | 415 160 936 | 27.3130 | 9.0694 |
| 747 | 55 80 09 | 416 832 723 | 27.3313 | 9.0735 |
| 748 | 55 95 04 | 418 508 992 | 27.3496 | 9.0775 |
| 749 | 56 10 01 | 420 189 749 | 27.3679 | 9.0816 |
| 750 | 56 25 00 | 421 875 000 | 27.3861 | 9.0856 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 751 | 56 40 01 | 423 564 751 | 27.4044 | 9.0896 |
| 752 | 56 55 04 | 425 259 008 | 27.4226 | 9.0937 |
| 753 | 56 70 09 | 426 957 777 | 27.4408 | 9.0977 |
| 754 | 56 85 16 | 428 661 064 | 27.4591 | 9.1017 |
| 755 | 57 00 25 | 430 368 875 | 27.4773 | 9.1057 |
| 756 | 57 15 36 | 432 081 216 | 27.4955 | 9.1098 |
| 757 | 57 30 49 | 433 798 093 | 27.5136 | 9.1138 |
| 758 | 57 45 64 | 435 519 512 | 27.5318 | 9.1178 |
| 759 | 57 60 81 | 437 245 479 | 27.5500 | 9.1218 |
| 760 | 57 76 00 | 438 976 000 | 27.5681 | 9.1258 |
| 761 | 57 91 21 | 440 711 081 | 27.5862 | 9.1298 |
| 762 | 58 06 44 | 442 450 728 | 27.6043 | 9.1338 |
| 763 | 58 21 69 | 444 194 947 | 27.6225 | 9.1378 |
| 764 | 58 36 96 | 445 943 744 | 27.6405 | 9.1418 |
| 765 | 58 52 25 | 447 697 125 | 27.6586 | 9.1458 |
| 766 | 58 67 56 | 449 455 096 | 27.6767 | 9.1498 |
| 767 | 58 82 89 | 451 217 663 | 27.6948 | 9.1537 |
| 768 | 58 98 24 | 452 984 832 | 27.7128 | 9.1577 |
| 769 | 59 13 61 | 454 756 609 | 27.7308 | 9.1617 |
| 770 | 59 29 00 | 456 533 000 | 27.7489 | 9.1657 |
| 771 | 59 44 41 | 458 314 011 | 27.7669 | 9.1696 |
| 772 | 59 59 84 | 460 099 648 | 27.7849 | 9.1736 |
| 773 | 59 75 29 | 461 889 917 | 27.8029 | 9.1775 |
| 774 | 59 90 76 | 463 684 824 | 27.8209 | 9.1815 |
| 775 | 60 06 25 | 465 484 375 | 27.8388 | 9.1855 |
| 776 | 60 21 76 | 467 288 576 | 27.8568 | 9.1894 |
| 777 | 60 37 29 | 469 097 433 | 27.8747 | 9.1933 |
| 778 | 60 52 84 | 470 910 952 | 27.8927 | 9.1973 |
| 779 | 60 68 41 | 472 729 139 | 27.9106 | 9.2012 |
| 780 | 60 84 00 | 474 552 000 | 27.9285 | 9.2052 |
| 781 | 60 99 61 | 476 379 541 | 27.9464 | 9.2091 |
| 782 | 61 15 24 | 478 211 768 | 27.9643 | 9.2130 |
| 783 | 61 30 89 | 480 048 687 | 27.9821 | 9.2170 |
| 784 | 61 46 56 | 481 890 304 | 28.0000 | 9.2209 |
| 785 | 61 62 25 | 483 736 625 | 28.0179 | 9.2248 |
| 786 | 61 77 96 | 485 587 656 | 28.0357 | 9.2287 |
| 787 | 61 93 69 | 487 443 403 | 28.0535 | 9.2326 |
| 788 | 62 09 44 | 489 303 872 | 28.0713 | 9.2365 |
| 789 | 62 25 21 | 491 169 069 | 28.0891 | 9.2404 |
| 790 | 62 41 00 | 493 039 000 | 28.1069 | 9.2443 |
| 791 | 62 56 81 | 494 913 671 | 28.1247 | 9.2482 |
| 792 | 62 72 64 | 496 793 088 | 28.1425 | 9.2521 |
| 793 | 62 88 49 | 498 677 257 | 28.1603 | 9.2560 |
| 794 | 63 04 36 | 500 566 184 | 28.1780 | 9.2599 |
| 795 | 63 20 25 | 502 459 875 | 28.1957 | 9.2638 |
| 796 | 63 36 16 | 504 358 336 | 28.2135 | 9.2677 |
| 797 | 63 52 09 | 506 261 573 | 28.2312 | 9.2716 |
| 798 | 63 68 04 | 508 169 592 | 28.2489 | 9.2754 |
| 799 | 63 84 01 | 510 082 399 | 28.2666 | 9.2793 |
| 800 | 64 00 00 | 512 000 000 | 28.2843 | 9.2832 |

| Number | Square. | Cube. | Square Root. | Cube Root |
|---|---|---|---|---|
| 801 | 64 16 01 | 513 922 401 | 28.3019 | 9.2870 |
| 802 | 64 32 04 | 515 849 608 | 28.3196 | 9.2909 |
| 803 | 64 48 09 | 517 781 627 | 28.3373 | 9.2948 |
| 804 | 64 64 16 | 519 718 464 | 28.3549 | 9.2986 |
| 805 | 64 80 25 | 521 660 125 | 28.3725 | 9.3025 |
| 806 | 64 96 36 | 523 606 616 | 28.3901 | 9.3063 |
| 807 | 65 12 49 | 525 557 943 | 28.4077 | 9.3102 |
| 808 | 65 28 64 | 527 514 112 | 28.4253 | 9.3140 |
| 809 | 65 44 81 | 529 475 129 | 28.4429 | 9.3179 |
| 810 | 65 61 00 | 531 441 000 | 28.4605 | 9.3217 |
| 811 | 65 77 21 | 533 411 731 | 28.4781 | 9.3255 |
| 812 | 65 93 44 | 535 387 328 | 28.4956 | 9.3294 |
| 813 | 66 09 69 | 537 367 797 | 28.5132 | 9.3332 |
| 814 | 66 25 96 | 539 353 144 | 28.5307 | 9.3370 |
| 815 | 66 42 25 | 541 343 375 | 28.5482 | 9.3408 |
| 816 | 66 58 56 | 543 338 496 | 28.5657 | 9.3447 |
| 817 | 66 74 89 | 545 338 513 | 28.5832 | 9.3485 |
| 818 | 66 91 24 | 547 343 432 | 28.6007 | 9.3523 |
| 819 | 67 07 61 | 549 353 259 | 28.6182 | 9.3561 |
| 820 | 67 24 00 | 551 368 000 | 28.6356 | 9.3599 |
| 821 | 67 40 41 | 553 387 661 | 28.6531 | 9.3637 |
| 822 | 67 56 84 | 555 412 248 | 28.6705 | 9.3675 |
| 823 | 67 73 29 | 557 441 767 | 28.6880 | 9.3713 |
| 824 | 67 89 76 | 559 476 224 | 28.7054 | 9.3751 |
| 825 | 68 06 25 | 561 515 625 | 28.7228 | 9.3789 |
| 826 | 68 22 76 | 563 559 976 | 28.7402 | 9.3827 |
| 827 | 68 39 29 | 565 609 283 | 28.7576 | 9.3865 |
| 828 | 68 55 84 | 567 663 552 | 28.7750 | 9.3902 |
| 829 | 68 72 41 | 569 722 789 | 28.7924 | 9.3940 |
| 830 | 68 89 00 | 571 787 000 | 28.8097 | 9.3978 |
| 831 | 69 05 61 | 573 856 191 | 28.8271 | 9.4016 |
| 832 | 69 22 24 | 575 930 368 | 28.8444 | 9.4053 |
| 833 | 69 38 89 | 578 009 537 | 28.8617 | 9.4091 |
| 834 | 69 55 56 | 580 093 704 | 28.8791 | 9.4129 |
| 835 | 69 72 25 | 582 182 875 | 28.8964 | 9.4166 |
| 836 | 69 88 96 | 584 277 056 | 28.9137 | 9.4204 |
| 837 | 70 05 69 | 586 376 253 | 28.9310 | 9.4241 |
| 838 | 70 22 44 | 588 480 472 | 28.9482 | 9.4279 |
| 839 | 70 39 21 | 590 589 719 | 28.9655 | 9.4316 |
| 840 | 70 56 00 | 592 704 000 | 28.9828 | 9.4354 |
| 841 | 70 72 81 | 594 823 321 | 29.0000 | 9.4391 |
| 842 | 70 89 64 | 596 947 688 | 29.0172 | 9.4429 |
| 843 | 71 06 49 | 599 077 107 | 29.0345 | 9.4466 |
| 844 | 71 23 36 | 601 211 584 | 29.0517 | 9.4503 |
| 845 | 71 40 25 | 603 351 125 | 29.0689 | 9.4541 |
| 846 | 71 57 16 | 605 495 736 | 29.0861 | 9.4578 |
| 847 | 71 74 09 | 607 645 423 | 29.1033 | 9.4615 |
| 848 | 71 91 04 | 609 800 192 | 29.1204 | 9.4652 |
| 849 | 72 08 01 | 611 960 049 | 29.1376 | 9.4690 |
| 850 | 72 25 00 | 614 125 000 | 29.1548 | 9.4727 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 851 | 72 42 01 | 616 295 051 | 29.1719 | 9.4764 |
| 852 | 72 59 04 | 618 470 208 | 29.1890 | 9.4801 |
| 853 | 72 76 09 | 620 650 477 | 29.2062 | 9.4838 |
| 854 | 72 93 16 | 622 835 864 | 29.2233 | 9.4875 |
| 855 | 73 10 25 | 625 026 375 | 29.2404 | 9.4912 |
| 856 | 73 27 36 | 627 222 016 | 29.2575 | 9.4949 |
| 857 | 73 44 49 | 629 422 793 | 29.2746 | 9.4986 |
| 858 | 73 61 64 | 631 628 712 | 29.2916 | 9.5023 |
| 859 | 73 78 81 | 633 839 779 | 29.3087 | 9.5060 |
| 860 | 73 96 00 | 636 056 000 | 29.3258 | 9.5097 |
| 861 | 74 13 21 | 638 277 381 | 29.3428 | 9.5134 |
| 862 | 74 30 44 | 640 503 928 | 29.3598 | 9.5171 |
| 863 | 74 47 69 | 642 735 647 | 29.3769 | 9.5207 |
| 864 | 74 64 96 | 644 972 544 | 29.3939 | 9.5244 |
| 865 | 74 82 25 | 647 214 625 | 29.4109 | 9.5281 |
| 866 | 74 99 56 | 649 461 896 | 29.4279 | 9.5317 |
| 867 | 75 16 89 | 651 714 363 | 29.4449 | 9.5354 |
| 868 | 75 34 24 | 653 972 032 | 29.4618 | 9.5391 |
| 869 | 75 51 61 | 656 234 909 | 29.4788 | 9.5427 |
| 870 | 75 69 00 | 658 503 000 | 29.4958 | 9.5464 |
| 871 | 75 86 41 | 660 776 311 | 29.5127 | 9.5501 |
| 872 | 76 03 84 | 663 054 848 | 29.5296 | 9.5537 |
| 873 | 76 21 29 | 665 338 617 | 29.5466 | 9.5574 |
| 874 | 76 38 76 | 667 627 624 | 29.5635 | 9.5610 |
| 875 | 76 56 25 | 669 921 875 | 29.5804 | 9.5647 |
| 876 | 76 73 76 | 672 221 376 | 29.5973 | 9.5683 |
| 877 | 76 91 29 | 674 526 133 | 29.6142 | 9.5719 |
| 878 | 77 08 84 | 676 836 152 | 29.6311 | 9.5756 |
| 879 | 77 26 41 | 679 151 439 | 29.6479 | 9.5792 |
| 880 | 77 44 00 | 681 472 000 | 29.6648 | 9.5828 |
| 881 | 77 61 61 | 683 797 841 | 29.6816 | 9.5865 |
| 882 | 77 79 24 | 686 128 968 | 29.6985 | 9.5901 |
| 883 | 77 96 89 | 688 465 387 | 29.7153 | 9.5937 |
| 884 | 78 14 56 | 690 807 104 | 29.7321 | 9.5973 |
| 885 | 78 32 25 | 693 154 125 | 29.7489 | 9.6010 |
| 886 | 78 49 96 | 695 506 456 | 29.7658 | 9.6046 |
| 887 | 78 67 69 | 697 864 103 | 29.7825 | 9.6082 |
| 888 | 78 85 44 | 700 227 072 | 29.7993 | 9.6118 |
| 889 | 79 03 21 | 702 595 369 | 29.8161 | 9.6154 |
| 890 | 79 21 00 | 704 969 000 | 29.8329 | 9.6190 |
| 891 | 79 38 81 | 707 347 971 | 29.8496 | 9.6226 |
| 892 | 79 56 64 | 709 732 288 | 29.8664 | 9.6262 |
| 893 | 79 74 49 | 712 121 957 | 29.8831 | 9.6298 |
| 894 | 79 92 36 | 714 516 984 | 29.8998 | 9.6334 |
| 895 | 80 10 25 | 716 917 375 | 29.9166 | 9.6370 |
| 896 | 80 28 16 | 719 323 136 | 29.9333 | 9.6406 |
| 897 | 80 46 09 | 721 734 273 | 29.9500 | 9.6442 |
| 898 | 80 64 04 | 724 150 792 | 29.9666 | 9.6477 |
| 899 | 80 82 01 | 726 572 699 | 29.9833 | 9.6513 |
| 900 | 81 00 00 | 729 000 000 | 30.0000 | 9.6549 |

| Number | Square. | Cube. | Square Root. | Cube Root. |
|---|---|---|---|---|
| 901 | 81 18 01 | 731 432 701 | 30.0167 | 9.6585 |
| 902 | 81 36 04 | 733 870 808 | 30.0333 | 9.6620 |
| 903 | 81 54 09 | 736 314 327 | 30.0500 | 9.6656 |
| 904 | 81 72 16 | 738 763 264 | 30.0666 | 9.6692 |
| 905 | 81 90 25 | 741 217 625 | 30.0832 | 9.6727 |
| 906 | 82 08 36 | 743 677 416 | 30.0998 | 9.6763 |
| 907 | 82 26 49 | 746 142 643 | 30.1164 | 9.6799 |
| 908 | 82 44 64 | 748 613 312 | 30.1330 | 9.6834 |
| 909 | 82 62 81 | 751 089 429 | 30.1496 | 9.6870 |
| 910 | 82 81 00 | 753 571 000 | 30.1662 | 9.6905 |
| 911 | 82 99 21 | 756 058 031 | 30.1828 | 9.6941 |
| 912 | 83 17 44 | 758 550 825 | 30.1993 | 9.6976 |
| 913 | 83 35 69 | 761 048 497 | 30.2159 | 9.7012 |
| 914 | 83 53 96 | 763 551 944 | 30.2324 | 9.7047 |
| 915 | 83 72 25 | 766 060 875 | 30.2490 | 9.7082 |
| 916 | 83 90 56 | 768 575 296 | 30.2655 | 9.7118 |
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